www.mtg.in | July 2016 | Pages 92 | ₹ 30

1 Cracking the JEE ADVANCED

Aman Bansal

JEE ADVANCED • WB JEE 2016 SOLVED PAPERS

MATHEMATICS today

India's #1 MATHEMATICS MONTHLY for JEE (Main & Advanced)

MATHS MUSING
10 GREAT PROBLEMS

Concept Boosters
Class XI - XII

MATH ARCHIVES

Monthly Practice Problems
(XI & XII)

ACE YOUR WAY
CBSE XI-XII SERIES 3

BRAIN @ WORK

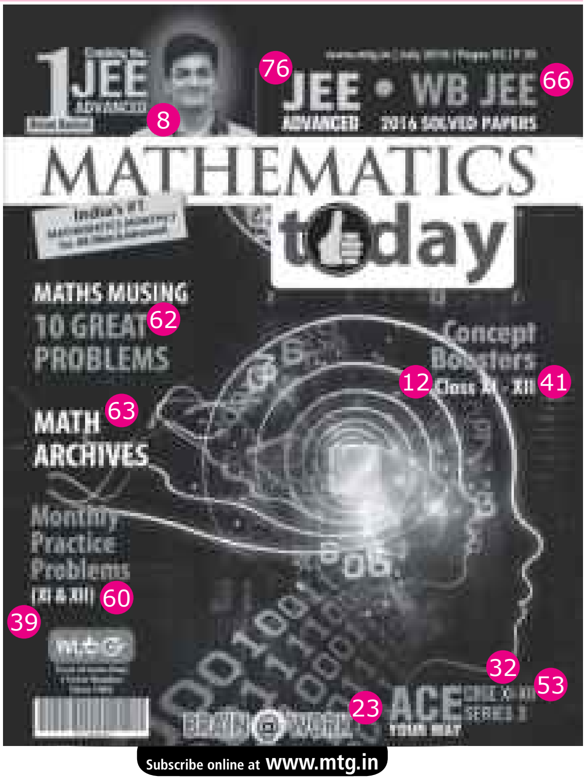
MATHEMATICS today

Vol. XXXIV No. 7 July 2016

Corporate Office:
Plot 99, Sector 44 Institutional Area,
Gurgaon -122 003 (HR), Tel : 0124-6601200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office:
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110029.
Managing Editor : Mahabir Singh
Editor : Anil Ahlawat

CONTENTS



Owned, Printed and Published by Mahabir Singh from 406, Taj Apartment, New Delhi - 29 and printed by Personal Graphics and Advertisers (P) Ltd., Okhla Industrial Area, Phase-II, New Delhi. Readers are advised to make appropriate thorough enquiries before acting upon any advertisements published in this magazine. Focus/Infocus features are marketing incentives MTG does not vouch or subscribe to the claims and representations made by advertisers. All disputes are subject to Delhi jurisdiction only.
Editor : Anil Ahlawat

GOOD NEWS FOR OUR READERS

We had earlier shared our intention to increase the price of this magazine and its sister magazines due to steep all round increase in costs. We have since been flooded with calls & mails requesting us not to increase the costs to benefit the student community at large and especially students from the economically weaker section.

We are happy to share that our editorial board has accepted the request in an endeavour to ensure that the largest circulating monthly magazines continue to be available to students at very reasonable costs.

Rvp

Cracking the JEE Advanced EXAM

Aman Bansal

JEE Advanced 2016 Topper Interview : How Aman Bansal scored 320 out of 372?

It was a 'dream come true' moment for Aman Bansal when he found out that the AIR 1 in JEE Advanced 2016 is none other than he himself. Coming from a middle class family where his father is an engineer and mother a homemaker, Aman's passion to be a Software Engineer kept him motivated to prepare well and get a high JEE Advanced rank which would enable him to choose his choice of IIT. With AIR 1 now, all admission options and choices are following this Jaipur boy who attributes his success to his systematic preparation strategy, parental support and exchange of knowledge among his peer group.

Aman, who was also among the high scorers in JEE Main and Class XII with 96.2% marks, shares how he planned his preparation and discusses his routine study schedule. He shares that he utilised his facebook account to become member of JEE Preparation Groups and also engaged in group studies to understand the level of competition and bring clarity on his doubts. He also talks on his hobbies, his favourite sports and more...

Congratulations for your outstanding performance in JEE Advanced 2016! What was your reaction upon knowing your rank?

Aman Bansal: Thank you. It came as a pleasant surprise for me. I was not expecting Rank 1 despite having performed well in the exam. I was hoping to be among top 10 rankers, but to be AIR 1 was beyond my expectations. It was like a dream come true and currently I am living the moment to the fullest!

What is your score in JEE Advanced 2016? How was your performance in JEE Main?

Aman Bansal: I have scored 320 out of 372. In JEE Main, my score was 323 out of a total of 360.

Tell us something about your family. How was support from your parents?

Aman Bansal: My family consists of my parents, grandmother and younger sister. My father is a government employee and my mother is a homemaker. My sister studies in Class VIII. All my family members are very supportive and motivated me throughout the preparation period. Also, there was no pressure from my parents, be it regarding studying engineering or scoring any rank or marks. They have always supported me irrespective of my performance.

When and how did you decide to study engineering? How did you start preparation?

Aman Bansal: I decided to pursue engineering after my Class X board exams. My decision was influenced by my love for Physics and Mathematics. Although my father is an Engineer, there was no compulsion from my family to follow suit. Right after my

Class X exam, I joined coaching and diligently followed the guidance of my mentors.

What was your preparation strategy and routine study period for JEE Advanced?

Aman Bansal: I had covered the syllabus during JEE Main preparation itself. For JEE Advanced, I focused on revision and reviewing my weaknesses. Along with revision I also devoted time in solving test papers and taking mock tests to brush up my exam taking skills and decide on the final exam day strategy. On an average, I used to spend 7 to 8 hours for studying. Some days I even stretched the study hours for 9 to 10 hours in case of special assignments or class or mock test preparation. Before 2 weeks to JEE Advanced exam, I reduced number of hours on studying and spent 1-2 hours daily to keep myself de-stressed and refreshed for the actual exam day.

Which subject was the easiest and which was the toughest according to you in the JEE Advanced? How did you find the difficulty level of exam?

Aman Bansal: For me, the easiest sections to attempt were Physics and Mathematics since these are also my favourite subjects. Chemistry was my weak area. The overall difficulty level of the exam for me was moderate to tough.

> **"Regular studies are very important. Utilize your time properly. While studying, underline and highlight the important parts. Discuss with your peers and faculty members to explore various problem solving techniques. Maintain a healthy competition with your peers. Do not get stressed and give your best shot with 100% confidence."**

Now that you are the topper of JEE Advanced, what do you think was the key factor behind your spectacular performance in the exam?

Aman Bansal: I would say my mentors and peers were the biggest factors behind my success. My mentors and guides at my coaching institute helped me immensely right from clarifying the concepts to providing the right study materials, test papers, analyzing my mistakes and suggesting ways of rectifying those errors and improving my performance. Secondly, my classmates gave me the idea of the competition I have. With regular interaction with fellow JEE aspirants I learnt various nuances of preparation, tips and tricks. I also went for group studies with them which were immensely helpful.

Do you think coaching is necessary to crack JEE Advanced exam?

Aman Bansal: For me, it is very necessary as without the right guidance aspirants are mostly clueless about the right preparation strategy. They may not get through the right study materials and mock tests. I studied in Allen Institute. Even if one opts to solve the test papers himself, getting them checked by mentors and following their feedback is very important.

It was quite a busy preparation schedule for you. Could you find time to connect with your friends during your preparation?

Aman Bansal: Yes of course, I took out some time from my study schedule to catch up with my friends. Since most of my friends were from my school and coaching institute, I did not have to give much effort to keep in touch with them. I even went out for a couple of outings with them to relax my mind. However, I made sure that I do not waste much time or strain myself during the process.

Are you active on Facebook or any other social media platform? Did you turn to any recreational activity before appearing for JEE Advanced exam?

Aman Bansal: I was active on Facebook and few other social media platforms as member of JEE Main and Advanced preparation groups and communities. I did not spend my time on chatting or other non-constructive activities in social media.

I love sports and play badminton regularly. Even before my exams, I used to play for a while to refresh myself. I am also very fond of playing indoor games. I believe that some recreational activity is definitely required as no one can study constantly without break.

Which is your preferred IIT and engineering branch?

Aman Bansal: I am aiming to take admission in IIT Bombay in Computer Science Engineering.

Please share your message for JEE aspirants who will be appearing for the entrance exam next year.

Aman Bansal: Regular studies are very important. Utilize your time properly. While studying, underline and highlight the important parts. Discuss with your peers and faculty members to explore various problem solving techniques. Maintain a healthy competition with your peers. Do not get stressed and give your best shot with 100% confidence.

Courtesy : careers360.com

CONCEPT BOOSTERS

Class XI

SEQUENCES & SERIES

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

SEQUENCE

A succession of numbers $a_1, a_2, a_3, ..., a_n, ...$ formed, according to some definite rule, is called a sequence.

ARITHMETIC PROGRESSION (A.P.)

A sequence of numbers $\{a_n\}$ is called an arithmetic progression, if there is a number d, such that $d = a_n - a_{n-1}$ for all n. The such number d is called the common difference (c.d.) of the A.P.

Useful Formulae

If a = first term, d = common difference and n is the number of terms, then

- n^{th} term is denoted by t_n and is given by $t_n = a + (n - 1)d$
- Sum of first n terms is denoted by S_n and is given by

  $$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

  where l is the last term in the series

  or $\quad l = t_n = a + (n - 1)d$
- If terms are given in A.P., and their sum is known, then the terms must be picked up in the following ways –
  - For three terms in A.P., we choose them as $(a - d), a, (a + d)$
  - For four terms in A.P., we choose them as $(a - 3d), (a - d), (a + d), (a + 3d)$
  - For five terms in A.P., we choose them as $(a - 2d), (a - d), a, (a + d), (a + 2d)$

  and so on.

Useful Properties

- If $t_n = an + b$, then the series so formed is an A.P.
- If $S_n = an^2 + bn$ then series so formed is an A.P.
- If every term of an A.P. is increased or decreased by some quantity, the resulting terms will also be in A.P.
- If every term of an A.P. is multiplied or divided by some non-zero quantity, the resulting terms will also be in A.P.
- In an A.P., the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.
- Sum and difference of corresponding terms of two A.P.'s will form an A.P.
- If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.
- If terms $a_1, a_2, ..., a_{2n-1}, a_{2n}$ are in A.P., then the sum of these terms will be equal to $(2n)\left(\frac{a_n + a_{n+1}}{2}\right)$.

GEOMETRIC PROGRESSION (G.P.)

A sequence of the numbers $\{a_n\}$, in which $a_1 \neq 0$, is called a geometric progression, if there is a number $r \neq 0$ such that $\frac{a_n}{a_{n-1}} = \text{constant} = r$ for all n, where r is called the common ratio (c.r.) of the G.P.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91).
He trains IIT and Olympiad aspirants.

Useful Formulae

If a = first term, r = common ratio and n is the number of terms, then

- n^{th} term, denoted by t_n and is given by $t_n = ar^{n-1}$
- Sum of first n terms denoted by S_n and is given by
$$S_n = \frac{a(1-r^n)}{1-r}, r<1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, r>1$$

In case $r = 1$, $S_n = na$.

- Sum of infinite terms (S_∞)
$$S_\infty = \frac{a}{1-r} \text{ (for } |r|<1 \text{ and } r \neq 0)$$
- If terms are given in G.P. and their product is known, then the terms must be picked up in the following ways –
  - For three terms in G.P., we choose them as $\frac{a}{r}$, a, ar.
  - For four terms in G.P., we choose them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
  - For five terms in G.P., we choose them as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

  and so on.

Useful Properties

- The product of the terms equidistant from the beginning and end is constant, and it is equal to the product of the first and the last term.
- If every term of a G.P. is multiplied or divided by the some non-zero fixed quantity, the resulting progression is also a G.P.
- If a_1, a_2, a_3 ... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ .... will also form a G.P. with common ratios r_1r_2 and $\frac{r_1}{r_2}$ respectively.
- If $a_1, a_2, a_3, ...$ be a G.P. of positive terms, then $\log a_1$, $\log a_2$, $\log a_3$, ... will be an A.P. and vice-versa.

HARMONIC PROGRESSION (H.P.)

A sequence is said to be in harmonic progression, if and only if the reciprocal of its terms form an arithmetic progression.

Useful Formulae

- If a, b are first two terms of an H.P. then
$$t_n = \frac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$$
- There is no formula for sum of n terms of an H.P.
- If terms are given in H.P., then the terms must be picked up in the following ways –
  - For three terms in H.P., we choose them as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$
  - For four terms in H.P., we choose them as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$
  - For five terms in H.P., we choose them as $\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$

  and so on.

Useful Properties

- If every term of a H.P. is multiplied or divided by some non-zero fixed quantity, the resulting progression is also a H.P.

INSERTION OF MEANS BETWEEN TWO NUMBERS

Arithmetic mean

- If three terms are in A.P., then the middle term is called the arithmetic mean (A.M.) between the other two *i.e.* if a, b, c are in A.P., then $b = \frac{a+c}{2}$ is the A.M. of a and c.
- If $a, A_1, A_2, ... A_n, b$ are in A.P., then $A_1, A_2, ..., A_n$ are called n A.M.'s between a and b. If d is the common difference, then $b = a + (n + 2 - 1)d \Rightarrow d = \frac{b-a}{n+1}$.

  Also, $A_i = a + id = a + i\frac{b-a}{n+1} = \frac{a(n+1-i)+ib}{n+1}$, $i = 1, 2, 3, ..., n$.

Note : The sum of n A.M's, is given by
$$A_1 + A_2 + ... + A_n = \frac{n}{2}(a+b)$$

Geometric mean

- If three terms are in G.P., then the middle term is called the geometric mean (G.M.) between the other two *i.e.*, if a, b, c are in G.P., then $b = \sqrt{ac}$ or $b = -\sqrt{ac}$ corresponding to a and both are positive or negative respectively is the G.M. of a and c.
- If $a, G_1, G_2 ... G_n, b$ are in G.P., then $G_1, G_2, ..., G_n$ are called n G.M.'s between a and b. If r is the common ratio, then
$$b = a \cdot r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$$

Also, $G_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}} \cdot b^{\frac{i}{n+1}}$, $i = 1, 2, ..., n$.

Note : The product of n G. M's is given by

$G_1 \cdot G_2 \cdot \ .... \ \cdot G_n = (\sqrt{ab})^n$

Harmonic mean

- If three terms are in H.P., then the middle term is called the harmonic mean (H.M.) between the other two, *i.e.*, if a, b, c are in H.P., then $\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \Rightarrow b = \frac{2ac}{a+c}$
- If $a, H_1, H_2 ... H_n, b$ are in H.P., then $H_1, H_2, ... , H_n$ are called n H.M.'s between a and b. If d is the common difference of the corresponding A.P., then

  $\frac{1}{b} = \frac{1}{a} + (n+2-1)d \Rightarrow d = \frac{a-b}{ab(n+1)}$

  Also, $\frac{1}{H_i} = \frac{1}{a} + id = \frac{1}{a} + i\frac{a-b}{ab(n+1)}$

  $\Rightarrow \quad H_i = \frac{ab(n+1)}{b(n-i+1)+ia}, \ i = 1, 2, 3, ..., n$

  Note : Term t_{n+1} is the arithmetic, geometric or harmonic mean of t_1 and t_{2n+1}, according as the terms t_1, t_{n+1} t_{2n+1} are in A.P., G.P. or H.P. respectively.

ARITHMETICO-GEOMETRIC SERIES

A series whose each term is formed, by multiplying corresponding terms of an A.P. and a G.P., is called an Arithmetico-Geometric series.

- **Summation of n terms of an Arithmetico-Geometric Series**

  Let $S_n = a + (a+d)r + (a+2d)r^2 + ...$
  $...+ [a + (n-1)d]r^{n-1}, \ d \neq 0, \ r \neq 0, 1$

  Multiply by 'r' and rewrite the series in the following way

  $rS_n = ar + (a+d)r^2 + (a+2d)r^3 + ...$
  $...+ [a + (n-2)d]r^{n-1} + [a + (n-1)d\ ]r^n$

  On subtraction,

  $S_n(1-r) = a + d(r + r^2 + ... + r^{n-1})$
  $- [a + (n-1)d]r^n$

  or, $\quad S_n(1-r) = a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d] \cdot r^n$

  or, $\quad S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]}{1-r} \cdot r^n$

- **Summation of Infinite Series**

  If $|r| < 1$, then $(n-1)r^n$, $r^{n-1} \to 0$, as $n \to \infty$

  Thus $S_\infty = S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

SUM OF MISCELLANEOUS SERIES

- **Difference Method :** Suppose $a_1, a_2, a_3, ...$ is a sequence such that the sequence $a_2 - a_1$, $a_3 - a_2, ...$ is either an A.P. or G.P., then the n^{th} term 'a_n' of this sequence is obtained as follows :

  $S = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$
  $S = \quad\quad a_1 + a_2 + ... + a_{n-2} + a_{n-1} + a_n$

  $\Rightarrow \quad a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + ...$
  $...+ (a_n - a_{n-1})]$

  Since the terms within the brackets are either in an A.P. or in a G.P., we can find the value of a_n, the n^{th} term. We can now find the sum of the n terms of the sequence as $S = \sum_{k=1}^{n} a_k$

- **$v_n - v_{n-1}$ Method :** Let $T_1, T_2, T_3 ....$ be the terms of a sequence, if there exists a sequence $v_1, v_2, v_3 ...$ satisfying $T_k = v_k - v_{k-1}, k \geq 1$,

  then $S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (v_k - v_{k-1}) = v_n - v_0$

INEQUALITIES

- **A.M. ≥ G.M. ≥ H.M.**

  Let $a_1, a_2, ..., a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + ... + a_n}{n}$,

  $G = (a_1 a_2 .... a_n)^{1/n}$ and

  $H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}\right)}$

  It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$

- **Weighted Means**

  Let $a_1, a_2, ..., a_n$ be n positive real numbers and w_1, $w_2, ..., w_n$ be n positive rational numbers. Then we define weighted arithmetic mean (A^*), weighted geometric mean (G^*) and weighted harmonic mean (H^*) as

  $A^* = \frac{a_1 w_1 + a_2 w_2 + ... + a_n w_n}{w_1 + w_2 + ... + w_n}$,

  $G^* = (a_1^{w_1} \cdot a_2^{w_2} ... a_n^{w_n})^{\frac{1}{w_1 + w_2 + ... w_n}}$

  and $H^* = \frac{w_1 + w_2 + ... + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + ... + \frac{w_n}{a_n}}$

$A^* \geq G^* \geq H^*$. Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$

Cauchy's Schwartz Inequality

- If $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are $2n$ real numbers, then
$(a_1b_1 + a_2b_2 + ... + a_n b_n)^2 \le (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$
with the equality holding if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$.

PROBLEMS

Single Correct Answer Type

1. The p^{th} term of an A.P. is a and q^{th} term is b, then the sum of its $(p + q)$ terms is

(a) $\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$ (b) $\frac{p-q}{2}\left[a+b-\frac{a-b}{p-q}\right]$

(c) $\frac{p+q}{2}\left[a-b+\frac{p-q}{a-b}\right]$ (d) $\frac{p-q}{2}\left[a+b+\frac{p-q}{a+b}\right]$

2. If $|x| < 1$ and $|y| < 1$, then $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \;... \infty$ is

(a) $\frac{x+y-xy}{(1-x)(1-y)}$ (b) $\frac{x-y-xy}{(1-x)(1-y)}$

(c) $\frac{x+y-xy}{(1+x)(1+y)}$ (d) $\frac{x-y-xy}{(1+x)(1+y)}$

3. In a centre test, there are p questions, in this $2^{(p-r)}$ students give wrong answers to at least r questions $(1 \le r \le p)$. If total number of wrong answers given is 2047, then the value of p is

(a) 14 (b) 13 (c) 12 (d) 11

4. If a, b, x, y are positive natural numbers such that $\frac{1}{x}+\frac{1}{y}=1$, then $\frac{a^x}{x}+\frac{b^y}{y}$ is

(a) $\le ab$ (b) $\ge ab$
(c) $= ab$ (d) can't be found out

5. The numbers $3^{2\sin 2\theta - 1}$, 14, $3^{4 - 2\sin 2\theta}$ are first three terms of an A.P. Its fifth term is equal to

(a) –25 (b) –12 (c) 40 (d) 53

6. The ratio between the sum of n terms of two A.P.'s is $3n + 8 : 7n + 15$. Then the ratio between their 12^{th} terms respectively is

(a) 5 : 7 (b) 7 : 16
(c) 12 : 11 (d) none of these

7. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+......$ upto $\infty = \frac{\pi^2}{6}$, then value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......$ upto to ∞ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$

8. If S be the sum, P be the product and R be the sum of the reciprocals of n terms of a G.P., then $\left(\frac{S}{R}\right)^n =$

(a) P (b) P^2 (c) P^3 (d) $\sqrt{P}$

9. Let $a_1, a_2, a_3, ...$ be in A.P. and a_p, a_q, a_r be in G.P. Then $a_q : a_p$ is equal to

(a) $\frac{r-p}{q-p}$ (b) $\frac{q-p}{r-q}$ (c) $\frac{r-q}{q-p}$ (d) 1

10. The sum of the two numbers is $2\frac{1}{6}$. An even numbers of arithmetic means are inserted between them and their sum exceeds their number by 1. Then the number of means inserted is

(a) 6 (b) 8 (c) 12 (d) 15

11. Given that $0<x<\frac{\pi}{4}$ and $\frac{\pi}{4}<y<\frac{\pi}{2}$ and $\sum_{k=0}^{\infty}(-1)^k \tan^{2k}x = p$; $\sum_{k=0}^{\infty}(-1)^k \cot^{2k}y = q$; then $\sum_{k=0}^{\infty}\tan^{2k}x\cot^{2k}y =$

(a) $\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}$ (b) $\frac{1}{\left\{\frac{1}{p}+\frac{1}{q}-\frac{1}{pq}\right\}}$

(c) $p + q - pq$ (d) $p + q + pq$

12. The sum of the series $\frac{8}{5}+\frac{16}{65}+......+\frac{128}{2^{18}+1}$ is

(a) $\frac{540}{1088}$ (b) $\frac{1088}{545}$ (c) $\frac{1001}{500}$ (d) $\frac{1013}{545}$

13. If $f(r)=1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{r}$ and $f(0)=0$, then value of $\sum_{r=1}^{n}(2r+1)f(r)=$

(a) $n^2 f(n)$

(b) $(n + 1)^2 f(n + 1) - \frac{n^2+3n+2}{2}$

(c) $(n + 1)^2 f(n) - \frac{n^2+n+1}{2}$

(d) $(n + 1)^2 f(n)$

Rvp

14. The n^{th} term of a series is given by $t_n = \frac{n^5+n^3}{n^4+n^2+1}$ and if sum of its n terms can be expressed as $S_n = a_n^2 + a + \frac{1}{b_n^2+b}$, where a_n and b_n are the n^{th} terms of some arithmetic progressions and a, b are some constants, then $\frac{b_n}{a_n}$ equal to

(a) $n\sqrt{2}$ (b) $\frac{n}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) 2

Multiple Correct Answer Type

15. The sum of the numerical series

$$\frac{1}{\sqrt{3}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{15}} + \ldots\ldots \text{ upto } n \text{ terms is}$$

(a) $\frac{\sqrt{3+4n}-\sqrt{3}}{4}$ (b) $\frac{n}{\sqrt{3+4n}+\sqrt{3}}$

(c) less than n (d) less than $\sqrt{n}/2$

16. The numbers $\frac{\sin x}{6}$, $\cos x$ and $\tan x$ will be in G.P. if

(a) $x = \frac{\pi}{3}$ (b) $x = \frac{5\pi}{6}$

(c) $x = \pm\frac{\pi}{3} + 2k\pi$ (d) $x = \pm\frac{\pi}{6} + 2k\pi$

17. If sum of n terms of an A.P. is given by $S_n = a + bn + cn^2$ where a, b, c are independent of n, then

(a) $a = 0$

(b) common difference of A.P. must be $2b$

(c) common difference of A.P. must be $2c$

(d) all of the above

18. Between two unequal numbers, if a_1, a_2 are two A.M.'s; g_1, g_2 are two G.M.'s and h_1, h_2 are two H.M.'s then $g_1 \cdot g_2$ is equal to

(a) a_1h_1 (b) a_1h_2 (c) a_2h_2 (d) a_2h_1

19. If positive numbers a, b, c, d are in harmonic progression and $a \neq b$, then

(a) $a + d > b + c$ is always true

(b) $a + b > c + d$ is always true

(c) $a + c > b + d$ always true

(d) $ad > bc$

20. If $(m + 1)^{th}$, $(n + 1)^{th}$ and $(r + 1)^{th}$ terms of an A.P. are in G.P. and m, n, r are in H.P., then the ratio of the first term of the A.P. to its common difference is

(a) $-\frac{n}{2}$ (b) $-\frac{m}{2}$ (c) r (d) $-\frac{mr}{m+r}$

21. If a, b, c are in H.P., then

(a) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(b) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$

(c) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.

(d) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

22. Let $S_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}$ and $T_n = 2 - \frac{1}{n}$, then

(a) $S_2 < T_2$

(b) If $S_k < T_k$ then $S_{k+1} < T_{k+1}$

(c) $S_n < T_n$ for all $n \geq 2$

(d) $S_n > T_n$ for all $n \geq 2007$

23. For a positive integer n, let

$$S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{2^n - 1}. \text{ Then,}$$

(a) $S_n \leq n$ (b) $S_n > n$

(c) $S_{2n} \leq n$ (d) $S_{2n} > n$

Comprehension Type

Paragraph for Q. 24 to 26

If x_1, x_2, ..., x_n are 'n' positive real numbers; then A.M. ≥ G.M. ≥ H.M.

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq (x_1 x_2 \ldots x_n)^{1/n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$$

equality occurs when numbers are same using this concept.

24. If $a > 0$, $b > 0$, $c > 0$ and the minimum value of $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2)$ is λabc, then λ is

(a) 1 (b) 2 (c) 3 (d) 6

25. If a, b, c, d, e, f are positive real numbers such that $a + b + c + d + e + f = 3$, then $x = (a + f)(b + e)(c + d)$ satisfies the relation

(a) $0 < x \leq 1$ (b) $1 \leq x \leq 2$

(c) $2 \leq x \leq 3$ (d) $3 \leq x \leq 4$

26. If a and b are two positive real numbers, and $a + b = 1$, then the greatest value of a^3b^4 is

(a) $\frac{3^2 4^3}{7^5}$ (b) $\frac{3^3 4^4}{7^7}$ (c) $\frac{7^7}{3^3 4^4}$ (d) $\frac{3^4 4^3}{7a}$

Matrix-Match Type

27. Match the value of x on the left with the value on the right.

	Column I		Column II
(A)	$5^2\ 5^4\ 5^6\ \ldots\ 5^{2x} = (0.04)^{-28}$	(p)	$3\log_3 5$
(B)	$x^2 = (0.2)^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)}$	(q)	4
(C)	$x = (0.16)^{\log_{5/2}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right)}$	(r)	2
(D)	$3^{x-1} + 3^{x-2} + 3^{x-3} + \ldots = 2\left(5^2+5+1+\frac{1}{5}+\frac{1}{5^2}+\ldots\right)$	(s)	7
		(t)	even integer

28. Let $a, b, c, p > 1$ and $q > 0$. Suppose a, b, c are in G.P.

	Column I		Column II
(A)	$\log_p a$, $\log_p b$, $\log_p c$ are in	(p)	G.P.
(B)	$\log_a p$, $\log_b p$, $\log_c p$ are in	(q)	A.G.P.
(C)	$a\log_p c$, $b\log_p b$, $c\log_p a$	(r)	H.P.
		(s)	A.P.

Integer Answer Type

29. Find the smallest natural number $m > 90$ for which $n = \underbrace{111\ldots\ldots1}_{m\text{ times}}$ is not a prime number. Hence find the value of $m - 87$.

30. Suppose a, x, y, z and b are in A.P. when $x + y + z = 15$, and $a, \alpha, \beta, \gamma, b$ are in H.P.. when $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{5}{3}$. Find a if $a > b$.

31. Find $\frac{8}{\pi}\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2k}{2+k^2+k^4}\right)$

32. If the lengths of the sides of a right angled triangle ABC right angled at C are in A.P., find $5(\sin A + \sin B)$.

33. A ball is dropped from a height of 900 cm. Each time it rebounds, it rises to 2/3 of the height it has fallen through. Find the two times of total distance travelled by the ball before it comes to rest in deca meters.

34. If $\log_x y$, $\log_z x$, $\log_y z$ are in G.P., $xyz = 64$ and x^3, y^3, z^3 are in A.P., find $x + y - z$.

35. If a_n denotes the coefficient of x^n in

$P(x) = (1 + x + 2x^2 + \ldots + 25x^{25})^2$, find $\frac{a_5}{5}$.

SOLUTIONS

1. (a) : Let x be the first term and d be the c.d of the A.P. Then,

$a = x + (p - 1)d,\ b = x + (q - 1)d$

$$\Rightarrow\ d = \frac{a-b}{p-q}$$

So, $x = a - \frac{(p-1)(a-b)}{p-q} = \frac{pb-qa+a-b}{p-q}$

Hence, $S_{p+q} = \frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$

2. (a) : The given sum

$S = (x + y) + (x^2 + xy + y^2) + \ldots$

$$= \frac{1}{(x-y)}\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots\}$$

$$= \frac{1}{(x-y)}\{(x^2+x^3+\ldots)-(y^2+y^3+\ldots)\}$$

$$= \frac{1}{(x-y)}\left(\frac{x^2}{1-x}-\frac{y^2}{1-y}\right) = \frac{1}{x-y}\left(\frac{x^2-y^2-x^2y+xy^2}{(1-x)(1-y)}\right)$$

$$= \frac{x+y-xy}{(1-x)(1-y)}$$

3. (d) : Number of students giving wrong answers to at least r questions $= 2^{p-r}$

Number of students giving wrong answers to at least $(r + 1)$ questions $= 2^{p-r-1}$

$\therefore$ Number of students giving wrong answers to exactly r questions $= 2^{p-r} - 2^{p-r-1}$.

Also number of students giving wrong answers to exactly p questions $= 2^{p-p} = 2^0 = 1$

$\therefore$ Total number of wrong answers

$$1(2^{p-1}-2^{p-2})+2(2^{p-2}-2^{p-3})+\ldots+(p-1)(2^1-2^0)+p(2^0)$$

$$= 2^{p-1}+(-2^{p-2}+2\cdot 2^{p-2})+(-2\cdot 2^{p-3}+3\cdot 2^{p-3})+\ldots + \{-(p-1)2^0+p\cdot 2^0\}$$

$= 2^{p-1} + 2^{p-2} + 2^{p-3} + \ldots + 2^0 = 2^p - 1$

$\Rightarrow 2^p - 1 = 2047 \Rightarrow 2^p = 2048 = 2^{11} \Rightarrow p = 11$

4. (b) : Consider the opposite numbers

$a^x, a^x, \ldots\ ky$ times and $b^y, b^y, \ldots\ kx$ times

$$\text{A.M.} = \frac{\{a^x+a^x+\ldots\ ky\text{ times}\}+\{b^y+b^y+\ldots\ kx\text{ times}\}}{kx+ky}$$

$$=\frac{kya^x+kxb^y}{k(x+y)}=\frac{ya^x+xb^y}{(x+y)} \quad \text{... (i)}$$

$$\text{G.M.}=\{(a^x\cdot a^x\text{..... } ky \text{ times})(b^y\cdot b^y\text{..... } kx \text{ times})\}^{\frac{1}{k(x+y)}}$$

$$=(a^{x(ky)}\cdot b^{y(kx)})^{\frac{1}{k(x+y)}}=(ab)^{\frac{kxy}{k(x+y)}}=(ab)^{\frac{xy}{(x+y)}} \quad \text{... (ii)}$$

As $\frac{1}{x}+\frac{1}{y}=1\Rightarrow\frac{x+y}{xy}=1$, *i.e.*, $x+y=xy$

$\therefore$ From (i) & (ii), $\frac{ya^x+xb^y}{xy}\ge ab$ or $\frac{a^x}{x}+\frac{b^y}{y}\ge ab$

5. **(d) :** Since the numbers are in A.P.

$\therefore\ 28 = 3^{2\sin 2\theta - 1} + 3^{4 - 2\sin 2\theta}$

or $28=\frac{9^{\sin 2\theta}}{3}+\frac{81}{9^{\sin 2\theta}}$

Put $x = 9 \sin 2\theta$, we get

$x^2 - 84x + 243 = 0$

or $(x - 81)(x - 3) = 0 \quad \Rightarrow \quad x = 81$ or 3

$\Rightarrow\ 9^{\sin 2\theta} = 81 \Rightarrow 3$ *i.e.*, 9^2 or $9^{1/2}$

$\therefore\ \sin 2\theta = 2$ or $1/2$

Since $\sin 2\theta$ cannot be greater than 1 so we choose $\sin 2\theta = 1/2$

Hence the terms in A.P. are 3^0, 14, 27 *i.e.* 1, 14, 27.

$\therefore\ T_5 = a + 4d = 1 + 4\cdot 13 = 53$

6. **(b) :** $\frac{S_n}{S'_n}=\frac{(n/2)[2a+(n-1)d]}{(n/2)[2a'+(n-1)d']}=\frac{3n+8}{7n+15}$

or $\frac{a+\left(\frac{n-1}{2}\right)d}{a'+\left(\frac{n-1}{2}\right)d'}=\frac{3n+8}{7n+15}$ ... (i)

Choosing $(n - 1)/2 = 11$ or $n = 23$ in (i), we get

$$\frac{T_{12}}{T'_{12}}=\frac{a+11d}{a'+11d'}=\frac{7}{16}$$

7. **(c) :** We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\text{........ upto }\infty$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\text{........ upto }\infty$$
$$-\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\text{.....upto }\infty\right]$$

$$=\frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

8. **(b) :** We have, $S = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$

$\Rightarrow\ S=\frac{a(1-r^n)}{1-r}$ .... (i)

$\therefore\ P$ = product $= a\cdot ar\cdot ar^2....ar^{n-1}$

$= a^n r^{1+2+3+4+5+....+n-1} = a^n\cdot r^{n(n-1)/2}$

$\therefore\ P^2 = a^{2n}\, r^{n(n-1)}$ .... (ii)

$$R=\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\frac{1}{ar^3}+....+\frac{1}{ar^{n-1}}$$

$\therefore\ R=\frac{1}{a}\times\frac{\left(1-\frac{1}{r^n}\right)}{1-(1/r)}=\frac{(r^n-1)}{(r-1)}\cdot\frac{1}{ar^{n-1}}$ .... (iii)

Now, $\frac{S}{R}=a\cdot\frac{(1-r^n)}{1-r}\cdot\frac{(r-1)}{(r^n-1)}\cdot ar^{n-1}=a^2r^{(n-1)}$

[by (i) and (iii)]

$\therefore\ (S/R)^n = a^{2n}\, r^{n(n-1)} = P^2$ [by (ii)]

9. **(c) :** Let common difference be d

$a_p = a_1 + (p - 1)d,\ a_q = a_1 + (q - 1)d$,

$a_r = a_1 + (r - 1)d$

As a_p, a_q, a_r are in G.P.

$\therefore\ \frac{a_q}{a_p}=\frac{a_r}{a_q}=\frac{a_q-a_r}{a_p-a_q}$ (by law of proportions)

or $\frac{a_q}{a_p}=\frac{a_r}{a_q}=\frac{a_1+(q-1)d-a_1-(r-1)d}{a_1+(p-1)d-a_1-(q-1)d}=\frac{q-r}{p-q}$

or $\frac{a_q}{a_p}=\frac{q-r}{p-q}=\frac{r-q}{q-p}$

10. **(c) :** Let the two numbers be a and b

Given $a+b=\frac{13}{6}$

and A.M.'s are A_1, A_2, ... A_{2n} inserted between a and b.

Here $a, A_1, A_2, ..., A_{2n}, b$ are in A.P. then given condition

$A_1 + A_2 + ... + A_{2n} = 2n + 1$

or $(a + A_1 + A_2 + ... + A_{2n} + b) - (a + b) = 2n + 1$

$\Rightarrow\ \frac{(2n+2)}{2}(a+b)-(a+b)=2n+1$

$\Rightarrow\ n(a + b) = 2n + 1 \Rightarrow 13n = 12n + 6 \Rightarrow n = 6$

Hence, number of means inserted is 12.

11. **(b) :** p = Infinite G.P. where $a = 1$, $r = -\tan^2 x$

$\therefore\ p=\frac{a}{1-r}=\frac{1}{1+\tan^2 x}=\cos^2 x$

Similarly, $q=\dfrac{1}{1+\cot^2 y}=\sin^2 y$

$$\therefore \quad S=\frac{1}{1-\tan^2 x\cot^2 y}=\frac{1}{1-\left(\dfrac{1-\cos^2 x}{\cos^2 x}\right)\left(\dfrac{1-\sin^2 y}{\sin^2 y}\right)}$$

$$\Rightarrow \quad S=\frac{pq}{p+q-1}=\frac{1}{\left\{\dfrac{1}{p}+\dfrac{1}{q}-\dfrac{1}{pq}\right\}}$$

12. (b) : Since,

$$S_n=\sum_{r=1}^{n}\frac{8r}{4r^4+1}=\sum_{r=1}^{n}\frac{8r}{(2r^2-2r+1)(2r^2+2r+1)}$$

$$=2\sum_{r=1}^{n}\frac{(2r^2+2r+1)-(2r^2-2r+1)}{(2r^2-2r+1)(2r^2+2r+1)}$$

$$=2\sum_{r=1}^{n}\left(\frac{1}{2r^2-2r+1}-\frac{1}{2r^2+2r+1}\right)$$

$$=2\left\{\frac{1}{1}-\frac{1}{5}+\frac{1}{5}-\frac{1}{13}+......+\frac{1}{2n^2-2n+1}-\frac{1}{2n^2+2n+1}\right\}$$

$$=2\left\{1-\frac{1}{2n^2+2n+1}\right\}=\frac{2(2n^2+2n)}{2n^2+2n+1}=\frac{4n^2+4n}{2n^2+2n+1}$$

$$\therefore \quad S_{16}=\frac{4(16)^2+4(16)}{2(16)^2+2(16)+1}=\frac{1088}{545}$$

13. (b) : Since $\displaystyle\sum_{r=1}^{n}(2r+1)f(r)$

$$=\sum_{r=1}^{n}(r^2+2r+1-r^2)f(r)=\sum_{r=1}^{n}\{(r+1)^2-r^2\}f(r)$$

$$=\sum_{r=1}^{n}\{(r+1)^2f(r)-(r+1)^2f(r+1)+(r+1)^2f(r+1)-r^2f(r)\}$$

$$=\sum_{r=1}^{n}(r+1)^2\{f(r)-f(r+1)\}+\sum_{r=1}^{n}\{(r+1)^2f(r+1)-r^2f(r)\}$$

$$=-\sum_{r=1}^{n}\frac{(r+1)^2}{(r+1)}+\sum_{r=1}^{n-1}(r+1)^2f(r+1)+(n+1)^2f(n+1)-\sum_{r=1}^{n}r^2f(r)$$

$$=-\sum_{r=1}^{n}(r+1)+\{2^2f(2)+3^2f(3)+...+n^2f(n)\}$$

$$+(n+1)^2f(n+1)-\{1^2f(1)+2^2f(2)+3^2f(3)+....+n^2f(n)\}$$

$$=-\sum_{r=1}^{n}r-\sum_{r=1}^{n}1+(n+1)^2f(n+1)-1^2f(1)$$

$$=\frac{-n(n+1)}{2}-n+(n+1)^2f(n+1)-f(1)$$

$$=(n+1)^2f(n+1)-\frac{n(n+3)}{2}-1$$

$$=(n+1)^2f(n+1)-\frac{(n^2+3n+2)}{2}$$

14. (d) : Since, $t_n=\dfrac{n^5+n^3}{n^4+n^2+1}=n-\dfrac{n}{n^4+n^2+1}$

$$=n+\frac{1}{2(n^2+n+1)}-\frac{1}{2(n^2-n+1)}$$

$\therefore$ Sum of n terms, $S_n=\displaystyle\sum_{n=1}^{n}t_n$

$$S_n=\sum_{n=1}^{n}n+\frac{1}{2}\left\{\sum_{n=1}^{n}\left(\frac{1}{n^2+n+1}-\frac{1}{n^2-n+1}\right)\right\}$$

$$=(1+2+3+.....+n)+\frac{1}{2}\left\{\frac{1}{3}-1+\frac{1}{7}-\frac{1}{3}+\frac{1}{13}-\frac{1}{7}+..... +\frac{1}{n^2+n+1}-\frac{1}{n^2-n+1}\right\}$$

$$=\frac{n(n+1)}{2}+\frac{1}{2}\left\{-1+\frac{1}{n^2+n+1}\right\}$$

$$=\frac{n^2}{2}+\frac{n}{2}-\frac{1}{2}+\frac{1}{2}\cdot\left(\frac{1}{n^2+n+1}\right)$$

$$=\left(\frac{n}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\right)^2-\frac{5}{8}+\frac{1}{\left(n\sqrt{2}+\dfrac{1}{\sqrt{2}}\right)^2+\dfrac{3}{2}}$$

But given $S_n=a_n^2+a+\dfrac{1}{b_n^2+b}$

On comparing, we get

$$a_n=\left(\frac{n}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\right),\ a=-\frac{5}{8},b_n=\left(n\sqrt{2}+\frac{1}{\sqrt{2}}\right),b=\frac{3}{2}$$

Hence, $\dfrac{b_n}{a_n}=2$, which is a constant.

15. (a, b, c) : In option (a), on rationalizing each term, we get series

$$=\frac{\sqrt{7}-\sqrt{3}}{7-3}+\frac{\sqrt{11}-\sqrt{7}}{11-7}+\frac{\sqrt{15}-\sqrt{11}}{15-11}+.....\text{upto } n \text{ terms}$$

$=\dfrac{1}{4}\left[\sqrt{3+4n}-\sqrt{3}\right]$ which is equal to $\dfrac{n}{\sqrt{3+4n}+\sqrt{3}}$

(c) Since $\frac{n}{\sqrt{3+4n}+\sqrt{3}}<n$, choice (c) is also correct

(d) $\frac{n}{\sqrt{3+4n}+\sqrt{3}}>\frac{n}{\sqrt{4n}}>\frac{\sqrt{n}}{2}$

16. (a, c) : $\frac{\sin x}{6}, \cos x, \tan x$ are in G.P.

$\Rightarrow \cos^2 x=\frac{\sin x \cdot \tan x}{6} \Rightarrow 6\cos^3 x + \cos^2 x - 1 = 0$

Put, $\cos x = t$, we get

$6t^3 + t^2 - 1 = 0 \Rightarrow (2t - 1)(3t^2 + 2t + 1) = 0$

As the quadratic factor has imaginary roots.

$\therefore$ $t = 1/2$ *i.e.*, $\cos x = 1/2 \Rightarrow x=\pm\frac{\pi}{3}+2k\pi, k\in Z$

17. (a, c) : $S_n=\frac{n}{2}[2a'+(n-1)d]=a+bn+cn^2$

$\Rightarrow na'+\frac{n(n-1)d}{2}=a+bn+cn^2$

$\Rightarrow \left(a'-\frac{d}{2}\right)n+\frac{n^2d}{2}=a+bn+cn^2$

On comparing the coefficients, we get

$a=0, b=a'-\frac{d}{2}, c=\frac{d}{2}$

18. (b, d) : Let A, a_1, a_2, B be in A. P.

$\therefore a_1=A+\frac{B-A}{3}=\frac{2A+B}{3}$

$\therefore a_2=A+2\cdot\frac{B-A}{3}=\frac{A+2B}{3}$

Also A, g_1, g_2, B are in G.P.

$\therefore \frac{B}{A}=r^3$

$\therefore g_1 = Ar = A(B/A)^{1/3}$

$g_2 = Ar^2 = A(B/A)^{2/3}$

$\therefore g_1g_2 = A^2(B/A) = AB$

Now, A, h_1, h_2, B are in H.P.

$\therefore \frac{1}{A},\frac{1}{h_1},\frac{1}{h_2},\frac{1}{B}$ are in A.P.

$\therefore \frac{1}{h_1}=\frac{1}{A}+\frac{1}{3}\left(\frac{1}{B}-\frac{1}{A}\right)=\frac{1}{A}+\frac{A-B}{3AB}$

$\Rightarrow h_1=\frac{3AB}{A+2B}$ and

$\frac{1}{h_2}=\frac{1}{A}+\frac{2}{3}\left(\frac{1}{B}-\frac{1}{A}\right)=\frac{3B+2(A-B)}{3AB}$

$\Rightarrow h_2=\frac{3AB}{2A+B}$

Obviously $g_1g_2 = AB = a_1h_2 = a_2h_1$

19. (a, d) : Take

$a=\frac{1}{p-3q}, b=\frac{1}{p-q}, c=\frac{1}{p+q}, d=\frac{1}{p+3q}$

Then $a + d > b + c$ easily follows

Since $(a + d) - (b + c)$

$=\frac{2p}{p^2-9q^2}-\frac{2p}{p^2-q^2}=2p\left[\frac{8q^2}{(p^2-9q^2)(p^2-q^2)}\right]$

which is positive ($\because$ $a, b, c, d > 0$)

Also $ad-bc=\frac{1}{p^2-9q^2}-\frac{1}{p^2-q^2}$

$=\frac{8q^2}{(p^2-9q^2)(p^2-q^2)}>0$

20. (a, d) : Given $(a + nd)^2 = (a + md)(a + rd)$

$\Rightarrow \left(\frac{a}{d}+n\right)^2=\left(\frac{a}{d}+m\right)\left(\frac{a}{d}+r\right)$ ... (i)

Also $n=\frac{2mr}{m+r} \Rightarrow mr=\frac{(m+r)n}{2}$ ... (ii)

Now from (i),

$\left(\frac{a}{d}\right)^2+2\left(\frac{an}{d}\right)+n^2=\left(\frac{a}{d}\right)^2+(m+r)\frac{a}{d}+mr$

$\Rightarrow \frac{a}{d}=\frac{n^2-mr}{m+r-2n}=\frac{n^2-\frac{(m+r)n}{2}}{m+r-2n}$ [from (ii)]

$\therefore \frac{a}{d}=-\frac{n}{2}=-\frac{mr}{m+r}$

21. (a, b, c, d) : $\because$ a, b, c are in H.P.

$\therefore \frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P.

$\Rightarrow \frac{a+b+c}{a},\frac{a+b+c}{b},\frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}$ are in A.P.

(Subtracting 1 from each term)

$\Rightarrow \frac{b+c}{a}-1,\frac{c+a}{b}-1,\frac{a+b}{c}-1$ are in A.P.

(Subtracting 1 from each term)

$\Rightarrow \frac{b+c-a}{a},\frac{c+a-b}{b},\frac{a+b-c}{c}$ are in A.P.

Also $b=\frac{2ac}{a+c}$

$\therefore \quad \frac{1}{b-a}+\frac{1}{b-c}=\frac{2b-(a+c)}{(b-a)(b-c)}=\frac{2b-(a+c)}{b^2-b(a+c)+ac}$

$=\frac{2b-(2ac/b)}{b^2-b(2ac/b)+ac}=\frac{2}{b}\cdot\frac{b^2-ac}{b^2-ac}=\frac{2}{b}$

22. (a, b, c) : $S_2=\frac{1}{1^2}+\frac{1}{2^2}=\frac{5}{4}, T_2=\frac{3}{2}$

$\Rightarrow \quad S_2<\frac{3}{2} \Rightarrow$ (a) is true

If $S_k < T_k$, then $\frac{1}{1^2}+\frac{1}{2^2}+........+\frac{1}{k^2}<2-\frac{1}{k}$

On adding $\frac{1}{(1+k)^2}$ on both sides, we get

$\frac{1}{1^2}+\frac{1}{2^2}+.......+\frac{1}{k^2}+\frac{1}{(1+k)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}$

Now $2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$ will be true if

$\frac{1}{k}-\frac{1}{(k+1)^2}>\frac{1}{k+1}$ or $(k+1)^2-k \ge k(k+1)$

or $k^2+k+1 \ge k^2+k$ which is true.

$\Rightarrow \quad S_{k+1} < T_{k+1}$

23. (a, d) : $S(n)=1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+....$

$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}+1}+...+\frac{1}{2^n-1}\right)$

$\le 1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+...$

$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+...+\frac{1}{2^{n-1}}\right)$

$= 1 + 1 + 1 + \ldots + 1$ (n terms) $= n$

Also $S(n)\ge 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+...$

$+\left(\frac{1}{2^{n-2}+1}+\frac{1}{2^{n-2}+2}+...+\frac{1}{2^{n-1}}\right)$

$>1+\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}(n-1 \text{ terms})=1+\left(\frac{n-1}{2}\right)=\frac{n+1}{2}$

$\therefore \quad S(2n)>\frac{2n+1}{2}=n+\frac{1}{2}>n$

24. (d) : As A.M. $\ge$ G.M.

$\frac{\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}}{6}\ge\left(\frac{c}{b}\times\frac{b}{c}\times\frac{c}{a}\times\frac{a}{c}\times\frac{a}{b}\times\frac{b}{a}\right)^{\frac{1}{6}}$

or $\frac{b^2+c^2}{bc}+\frac{c^2+a^2}{ca}+\frac{a^2+b^2}{ab}\ge 6$

or $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\ge 6abc$

$\therefore$ Minimum value of

$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$ is $6abc$

According to question $\lambda abc = 6abc \quad \therefore \quad \lambda = 6$

25. (a) : As a, b, c, d, e, f, are positive

$\therefore \quad (a+f)(b+e)(c+d)>0$. So, $x>0$ ....(i)

As A.M. $\ge$ G.M.

$\therefore \quad \frac{(a+f)+(b+e)+(c+d)}{3}\ge[(a+f)(b+e)(c+d)]^{1/3}$

or $\sqrt[3]{x}\le\frac{3}{3}$ or $x\le 1$ .... (ii)

From (i) and (ii), $0 < x \le 1$

26. (b) : By weighted mean

$\frac{3\left(\frac{a}{3}\right)+4\left(\frac{b}{4}\right)}{7}\ge\sqrt[7]{\left(\frac{a}{3}\right)^3\left(\frac{b}{4}\right)^4}$ or $\frac{a+b}{7}\ge\sqrt[7]{\frac{a^3}{3^3}\times\frac{b^4}{4^4}}$

27. A-s; B-r,t; C-q,t; D-p

(A) $5^{2+4+6+....+2x}=(25)^{28} \Rightarrow 5^{x(x+1)}=5^{56}$

$\Rightarrow \quad x^2+x-56=0 \quad \Rightarrow \quad x=7$

(B) $2\log_5 x=\log_{\sqrt{5}}\left(\frac{1/4}{1-1/2}\right)\log_5(0.2)$

$=\log_{\sqrt{5}}\left(\frac{1}{2}\right)\log_5\left(\frac{1}{5}\right)=-\frac{\log_5\left(\frac{1}{2}\right)}{\log_5\sqrt{5}}=\log_5 4 \Rightarrow x=2$

(C) $\log x=\log_{5/2}\left(\frac{1/3}{1-1/3}\right)\log(0.16)$

$= \log5/2\ (1/2) \log (2/5)^2 = \log 4 \Rightarrow x = 4$

(D) $3^x\left(\frac{1/3}{1-1/3}\right)=\frac{2(5^2)}{1-1/5} \quad \Rightarrow \quad \frac{1}{2}(3^x)=\frac{1}{2}(5^3)$

$\Rightarrow \quad x = 3\log_3 5$

28. A-s; B-r; C-q

$b^2 = ac \Rightarrow 2\log b = \log a + \log c$

$\Rightarrow \quad \log a, \log b, \log c$ are in A. P.

(A) $\log_p a, \log_p b, \log_p c$ are in A.P.

(B) Since $\frac{1}{\log_p a}, \frac{1}{\log_p b}, \frac{1}{\log_p c}$ are in H.P.

$\therefore \quad \log_a p, \log_b p, \log_c p$ are in H.P.

(C) Since a, b, c in G.P. and $\log_p c, \log_p b, \log_p a$ are in A.P.

$\therefore \quad a\log_p c, b\log_p c, c\log_p a$ are in A.G.P

29. (4) : $n = 1111 \ldots 1$(91 times)

$= 1 + 10 + 10^2 + \ldots + 10^{90}$

$$=\frac{10^{91}-1}{10-1}=\frac{10^{91}-1}{10^7-1}\cdot\frac{10^7-1}{10-1}=\frac{(10^7)^{13}-1}{10^7-1}\cdot\frac{10^7-1}{10-1}$$

$= (10^{84} + 10^{77} + 10^{70} + ... + 10^7 + 1) \times (10^6 + 10^5 + 10^4 + ... + 10 + 1)$

= Product of two integers

$\therefore$ n is not a prime number

$\therefore$ $m = 91 \Rightarrow m - 87 = 4$

30. (9) : Let $a = A - 2d$, $x = A - d$, $y = A$, $z = A + d$, $b = A + 2d$

Giving $x + y + z = 15 \Rightarrow A = 5 \Rightarrow a = 5 - 2d$, $b = 5 + 2d$

Also $\frac{1}{a},\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\frac{1}{b}$ are in A.P

$$\Rightarrow \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{3}{2}\left[\frac{1}{a}+\frac{1}{b}\right]=\frac{5}{3}$$

or $\frac{3}{2}\left(\frac{1}{5-2d}+\frac{1}{5+2d}\right)=\frac{5}{3} \Rightarrow d=\pm 2$

Take $d = -2$ since $a > b$. Hence $a = 9$

31. (2) : $\sum_{k=1}^{n}\tan^{-1}\frac{2k}{1+(k^2+k+1)(k^2-k+1)}$

$$\Rightarrow \sum_{k=1}^{n}\tan^{-1}\frac{(k^2+k+1)-(k^2-k+1)}{1+(k^2+k+1)(k^2-k+1)}$$

$$\Rightarrow \sum_{k=1}^{n}\{\tan^{-1}(k^2+k+1)-\tan^{-1}(k^2-k+1)\}$$

$= \tan^{-1}(n^2 + n + 1) - \tan^{-1}1$

When $n \to \infty$, then $\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{2k}{2+k^2+k^4}\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

Hence, $\frac{8}{\pi}\times\frac{\pi}{4}$ is 2

32. (7) : Here $\angle C = 90°$, $\angle A + \angle B = 90°$

$c^2 = a^2 + b^2$ and $2b = a + c$

Since $c = 2b - a$ and $c^2 = a^2 + b^2$

$$\Rightarrow (2b-a)^2 = a^2 + b^2 \quad \text{or} \quad \frac{b}{a}=\frac{4}{3}$$

$$\Rightarrow \frac{\sin B}{\sin A}=\frac{4}{3} \text{ or } \frac{\sin B+\sin A}{\sin B-\sin A}=\frac{7}{1} \text{ or } \cot\frac{B-A}{2}=\frac{7}{1}$$

$$\Rightarrow \cos\frac{A-B}{2}=\frac{7}{5\sqrt{2}}$$

Also $5(\sin A + \sin B) = 5\sqrt{2}\cos\left(\frac{A-B}{2}\right)=7$

33. (9) : According to question, total distance

$$=h+2\times\frac{2}{3}h+2\times\left(\frac{2}{3}\right)^2h+2\times\left(\frac{2}{3}\right)^3h+\text{... up to }\infty$$

$$=h+2\times\frac{2}{3}h(3)=5h=4500 \text{ cm}$$

$\Rightarrow$ $10h = 9000$ cm $= 9$ deca metres

34. (4) : According to question,

$$\frac{\log_z x}{\log_x y}=\frac{\log_y z}{\log_z x} \Rightarrow (\log x)^3=(\log z)^3 \Rightarrow x = z$$

Since $2y^3 = x^3 + z^3 \Rightarrow x^3 = y^3$ or $x = y$

Given $xyz = 64$ and $x = y = z$

$\therefore$ $x = y = z = 4$ and $x + y - z = 4$

35. (6) : $(1 + x + 2x^2 + 3x^3 + ... + 25x^{25})$ $(1 + x + 2x^2 + 3x^3 + ... + 25x^{25})$

$a_5 =$ coeff of $x^5 = 1{\cdot}5 + 1{\cdot}4 + 2{\cdot}3 + 3{\cdot}2 + 4{\cdot}1 + 5 = 30$

$$\Rightarrow \frac{a_5}{5}=6$$

BRAIN@WORK

COMPLEX NUMBERS

This article is a collection of shortcut methods, important formulas and MCQ's along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PET's.

IMPORTANT POINTS

(i) The sum of four consecutive powers of i is zero i.e. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \ \forall \ n \in N$.

(ii) The value of different integral powers of i are 1 or i or -1 or $-i$.

(iii) For any two real numbers $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either positive or zero. In other words, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is not valid if a and b both are negative.

(iv) For any positive real number a, we have $\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a}$

EQUALITY OF COMPLEX NUMBERS

(i) Two complex numbers z_1 and z_2 are said to be equal if and only if their real parts and imaginary parts are separately equal
i.e., $a + ib = c + id \Leftrightarrow a = c$ and $b = d$
or $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

(ii) Inequality relation (order relation) does not hold in case of complex numbers having non-zero imaginary parts. Hence it cannot be decided that out of the two complex numbers which one is greater or smaller. For example, the statement $9 + 6i > 3 + 2i$ makes no sense.

(iii) $a + ib > c + id$, it is possible only when $a > c$ and $b = d = 0$.

CONJUGATE OF A COMPLEX NUMBER

The conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined as $\bar{z} = \overline{x + iy} = x - iy$.

The conjugate of a complex number is obtained by just changing the sign of imaginary part of the complex number.

Properties of Conjugate

(i) $\bar{z}$ is the mirror image of z about real axis or x-axis.

(ii) $\overline{(\bar{z})} = z$

(iii) $z = \bar{z} \Leftrightarrow z$ is purely real.

(iv) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary.

(v) $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$

(vi) $\text{Im}(z) = -\text{Im}(\bar{z}) = \dfrac{z - \bar{z}}{2i}$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(x) $\overline{az_1 + bz_2} = a\bar{z}_1 + b\bar{z}_2$, where $a, b \in R$

(xi) $\overline{az_1 + bz_2} = \bar{a} \cdot \bar{z}_1 + \bar{b} \cdot \bar{z}_2$, where $a, b \in C$

(xii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

(xiii) $\text{Re}\,(z_1\bar{z}_2) = \text{Re}(\bar{z}_1 z_2) = \dfrac{1}{2}(z_1\bar{z}_2 + \bar{z}_1 z_2)$

(xiv) $\text{Im}(z_1\bar{z}_2) = -\text{Im}(\bar{z}_1 z_2) = \dfrac{1}{2i}(z_1\bar{z}_2 - \bar{z}_1 z_2)$

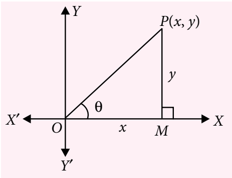
(xv) $\overline{(z^n)} = (\bar{z})^n$

(xvi) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

Sanjay Singh Mathematics Classes, Chandigarh

MODULUS OF A COMPLEX NUMBER

The modulus of a complex number $z = x + iy$ is denoted by $|z|$ and is defined as $|z|$ = non-negative square root of $(x^2 + y^2)$, *i.e.*, $|z| = \sqrt{x^2 + y^2}$



Geometrically, modulus of a complex number is the distance of the point (x, y) from the origin in the xy plane.

Properties of Modulus

(i) $|z| \ge 0 \Rightarrow \begin{cases} |z| = 0 \text{ iff } z = 0 \\ |z| > 0 \text{ iff } z \ne 0. \end{cases}$

(ii) $-|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$.

(iii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ (iv) $|z_1 z_2| = |z_1|\,|z_2|$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|};\ z_2 \ne 0$

(vi) $|z^n| = |z|^n$ (vii) $z\bar{z} = |z|^2$

(viii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)\,(\bar{z}_1 \pm \bar{z}_2)$
$= |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2) = |z_1|^2 + |z_2|^2 \pm 2\text{Re}\,(z_1\bar{z}_2)$

(ix) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1|\,|z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$
Geometrical significance of the given result is that in a parallelogram, the sum of the squares of the diagonals is twice the sum of the squares of the adjacent sides.

(xi) If z is unimodular, then $|z| = 1$. In case of a unimodular complex number z is taken as
$z = \cos\theta + i\sin\theta,\ \theta \in R$

(xii) $|z_1 \pm z_2| \le |z_1| + |z_2|$. In general
$|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \le |z_1| + |z_2| + |z_3| + \ldots + |z_n|$.

(xiii) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$.
Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.

PRINCIPAL VALUE OF ARG (z)

There are infinite number of values of θ satisfying the two equations $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$ and $\sin\theta = \dfrac{y}{\sqrt{x^2 + y^2}}$ simultaneously.

But there will be a unique value of θ such that $-\pi < \theta \le \pi$. Such value of argument (θ) is called the principal value of the argument.

Note :

(i) Argument of the complex number 0 is not defined.

(ii) $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and $\arg z_1 = \arg z_2$.

(iii) If $\arg(z) = \dfrac{\pi}{2}$ or $-\dfrac{\pi}{2}$, z is purely imaginary.
i.e., $z + \bar{z} = 0$

(iv) If $\arg(z) = 0$ or π, z is purely real. *i.e.*, $z = \bar{z}$

Properties of Argument (Amplitude)

The following properties are valid for general values of arguments:

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi,\ k \in I$
In general, $\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n)$

(ii) $\arg(z^n) = n \arg z + 2k\pi,\ k \in I$

(iii) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi,\ k \in I$

The following properties are valid for principal values of arguments :

(i) $\arg\bar{z} = -\arg z$ (ii) $\arg(z\bar{z}) = 0$

(iii) $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg z$

PROPERTIES OF CIS (θ) OR $e^{i\theta}$

(i) $\text{cis}\,(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$

(ii) $\text{cis}\,(-\theta) = \cos\theta - i\sin\theta = e^{-i\theta}$

(iii) $\text{cis}\,\theta + \text{cis}\,(-\theta) = e^{i\theta} + e^{-i\theta} = 2\cos\theta$

(iv) $\text{cis}\,\theta - \text{cis}\,(-\theta) = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

(v) $\text{cis}\theta_1 \cdot \text{cis}\theta_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} = \text{cis}(\theta_1 + \theta_2)$

(vi) $\dfrac{\text{cis}\,\theta_1}{\text{cis}\,\theta_2} = \dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1 - \theta_2)} = \text{cis}(\theta_1 - \theta_2)$

(vii) $\dfrac{1}{\text{cis}\,\theta} = \dfrac{1}{e^{i\theta}} = \text{cis}(-\theta)$

(viii) $(\text{cis}\,\theta)^n = (e^{i\theta})^n = \text{cis}(n\theta) = \cos n\theta + i\sin n\theta$

(ix) $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n$
$= \cos(n\pi/2 - n\theta) + i\sin(n\pi/2 - n\theta)$.

(x) $(\cos\theta + i\sin\phi)^n \ne \cos n\theta + i\sin n\phi$.

(xi) $\prod_{r=1}^{n}[\cos\theta_r + i\sin\theta_r] = \cos\left(\sum_{r=1}^{n}\theta_r\right) + i\sin\left(\sum_{r=1}^{n}\theta_r\right)$

(xii) $\prod_{r=1}^{n}(\cos r\theta + i\sin r\theta) = \prod_{r=1}^{n} e^{ir\theta}$

$$= \cos\left(\sum_{r=1}^{n} r\right)\theta + i\sin\left(\sum_{r=1}^{n} r\right)\theta$$

$$= \cos\frac{n(n+1)}{2}\theta + i\sin\frac{n(n+1)}{2}\theta$$

SQUARE ROOT OF A COMPLEX NUMBER

The square roots of $z = a + ib$ are

$$\pm\left[\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right] \text{for } b>0, \text{and}$$

$$\pm\left[\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right] \text{for } b<0$$

CUBE ROOTS OF UNITY

Cube roots of unity are

$$1,\ \omega=\frac{-1+i\sqrt{3}}{2} \text{ and } \omega^2=\frac{-1-i\sqrt{3}}{2}$$

Properties of Cube Roots of Unity

(i) $1 + \omega + \omega^2 = 0$ (ii) $\omega^3 = 1$

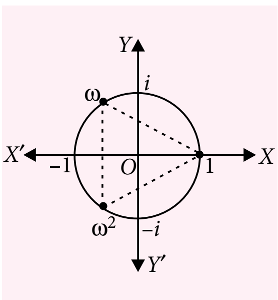
(iii) $\omega^{3n} = 1$, $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$

(iv) $1^p+\omega^p+(\omega^2)^p=\begin{cases}0, & \text{if } p \text{ is not a multiple of 3}\\ 3, & \text{if } p \text{ is a multiple of 3}\end{cases}$

(v) If a is any (+)ve number, then $a^{1/3}$ has roots $a^{1/3}(1)$, $a^{1/3}(\omega)$, $a^{1/3}(\omega^2)$.

If a is any (–)ve number, then $a^{1/3}$ has roots $-|a|^{1/3}$, $-|a|^{1/3}\omega$, $-|a|^{1/3}\omega^2$.

(vi) The cube roots of unity when represented on complex plane represents the vertices of an equilateral triangle inscribed in a unit circle, having centre at the origin and with one vertex being on positive real axis.



THE n^{th} ROOTS OF UNITY

The equation $x^n = 1$ has n roots which are called as the n^{th} roots of unity.

$\therefore\ x^n = 1 = \cos 0 + i \sin 0 = \cos 2k\pi + i \sin 2k\pi,\ k \in I$

$$\Rightarrow\ x = (\cos 2k\pi + i \sin 2k\pi)^{1/n} = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$$

$$=\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^k, \text{ where } k = 0, 1, 2, 3, ..., (n-1)$$

Let $\alpha=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}=e^{i\left(\frac{2\pi}{n}\right)}$

Then the n^{th} roots of the unity are α^k, where $k = 0, 1, 2, 3, ..., (n-1)$, *i.e.*, The n^{th} roots of unity are $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ which are in G.P.

Properties of n^{th} Roots of Unity

(i) Sum of n^{th} roots of unity is always zero.

i.e., $1 + \alpha + \alpha^2 + ... + \alpha^{n-1}$

$$=\frac{1-\alpha^n}{1-\alpha}=\frac{1-(\cos 2\pi+i\sin 2\pi)}{1-\alpha}=0$$

(ii) Sum of p^{th} powers of n^{th} roots of unity is zero, if p is not a multiple of n.

i.e., $1^p + \alpha^p + (\alpha^2)^p + ... + (\alpha^{n-1})^p = 0$ or $\sum_{r=0}^{n-1}(\alpha^r)^p=0$

(iii) Sum of p^{th} powers of n^{th} roots of unity is n, if p is a multiple of n.

i.e., $1^p + \alpha^p + (\alpha^2)^p + ... + (\alpha^{n-1})^p = n$ or $\sum_{r=0}^{n-1}(\alpha^r)^p=n$

(iv) Product of n^{th} roots of unity is $(-1)^{n-1}$

i.e., $1\cdot\alpha\cdot\alpha^2\cdot\alpha^3 \dots \alpha^{n-1}=\prod_{k=0}^{n-1}\alpha^k$

$$=\prod_{k=0}^{n-1}\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right)=(-1)^{n-1}$$

(v) $x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)\dots(x-\alpha^{n-1})$

(vi) If n^{th} roots of unity are plotted on the argand plane then they are representing the vertices of a regular plane polygon of n sides inscribed in a circle of radius one having centre at origin and one vertex being on positive real axis.

LOGARITHM OF A COMPLEX NUMBER $a + ib$

$\ln(a + ib) = \ln|z| + i(\arg z)$, where $z = a + ib$ and $(\arg z)$ is the principal argument.

e.g., $\ln(1 - i) = \ln|1 - i| + i\arg(1 - i) = \ln\sqrt{2} + i(-\pi/4)$

DISTANCE FORMULA

Distance between $A(z_1)$ and $B(z_2)$ is given by $AB = |z_2 - z_1|$.

SECTION FORMULA

(i) If a line segment joining the points $A(z_1)$ and $B(z_2)$ is divided by point $P(z)$ in the ratio $m_1 : m_2$ internally, then

$$z=\frac{m_1z_2+m_2z_1}{m_1+m_2}, m_1 \text{ and } m_2 \text{ are real}.$$

(ii) If a line segment joining the points $A(z_1)$ and $B(z_2)$ is divided by point $P(z)$ in the ratio $m_1 : m_2$ externally

$$\text{then } z=\frac{m_1z_2-m_2z_1}{m_1-m_2}, m_1 \text{ and } m_2 \text{ are real}.$$

(iii) If z bisects the join of z_1 and z_2, then $z=\frac{z_1+z_2}{2}$.

STRAIGHT LINES

(i) Equation of the line passing through the points z_1 and z_2 is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \text{ or } \frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

(ii) Equation of the line joining z_1 and z_2 in parametric form is given by $z = tz_1 + (1 - t)z_2$ where 't' is a parameter.

(iii) General equation of any line in Argand plane is of the form $a\bar{z} + \bar{a}z + b = 0$, where '$a$' is a fixed non-zero complex number and b is a fixed real number.

(iv) Points z_1, z_2, z_3 are collinear iff

(a) $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(b) $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real

(c) $\arg(z_2 - z_1) = \arg(z_3 - z_1)$

COMPLEX SLOPE OF THE LINE SEGMENT JOINING TWO POINTS

- If A and B are represented by unequal complex numbers z_1 and z_2 in the Argand plane, then the complex slope of AB is defined as $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$ and denoted by ω.

If ω_1 and ω_2 are the complex slopes of two lines in the Argand plane, then the lines are

(i) Perpendicular, if $\omega_1 + \omega_2 = 0$

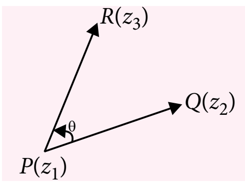
(ii) Parallel, if $\omega_1 = \omega_2$

- Complex slope of the line $a\bar{z} + \bar{a}z + b = 0$ is $\dfrac{-a}{\bar{a}} = -\dfrac{\text{Coeff. of } \bar{z}}{\text{Coeff. of } z}$

ANGLE BETWEEN THE TWO LINES

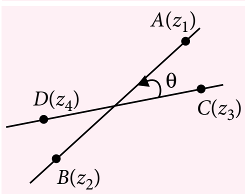
(i) Angle between the rays PR and PQ is

$$\theta = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$



(ii) Angle between the lines joining $A(z_1)$ and $B(z_2)$ and the line joining $C(z_3)$ and $D(z_4)$ is given by

$$\theta = \arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$$



(a) If AB coincides with CD, then

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = 0 \text{ or } \pi$$

$$\Rightarrow \left(\frac{z_1 - z_2}{z_3 - z_4}\right) \text{ is real } \Rightarrow \frac{z_1 - z_2}{z_3 - z_4} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4}$$

It follows that if $\dfrac{z_1 - z_2}{z_3 - z_4}$ is real, then the points A, B, C and D are collinear.

(b) If AB is perpendicular to CD, then

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

$\Rightarrow \dfrac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary.

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_4} + \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4} = 0$$

It follows that if $z_1 - z_2 = \pm k(z_3 - z_4)$, where k is purely imaginary number, then AB and CD are perpendicular to each other.

Equation of a Line Parallel and Perpendicular to $a\bar{z} + \bar{a}z + b = 0$ is

(i) The line parallel to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} + \bar{a}z + \lambda = 0$, where $\lambda \in R$.

(ii) The equation of a line perpendicular to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} - \bar{a}z + i\lambda = 0$, where $\lambda \in R$.

(iii) The length of the perpendicular from a point $P(z_0)$ to the line $a\bar{z} + \bar{a}z + b = 0$ is $\dfrac{|a\bar{z}_0 + \bar{a}z_0 + b|}{2|a|}$

CIRCLE

(i) The equation of a circle whose centre is at point having affix z_0 and radius r is $|z - z_0| = r$.

(ii) The equation of a circle described on a line segment joining $A(z_1)$ and $B(z_2)$ as diameter is $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$ or $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$

(iii) Equation of the circle passing through three non-collinear points z_1, z_2, z_3 is

$$\left(\frac{z - z_1}{\bar{z} - \bar{z}_1}\right)\left(\frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3}\right) = \left(\frac{z - z_2}{\bar{z} - \bar{z}_2}\right)\left(\frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}\right)$$

(iv) Four points z_1, z_2, z_3 and z_4 are concyclic if and only if $\dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is purely real.

(v) General equation of the circle in the Argand plane is of the form $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where a is fixed complex number and $b \in R$ whose centre is at $-a$ and radius $= \sqrt{|a|^2 - b}$.

(a) Circle will be a Real Circle if $|a|^2 > b$

(b) Circle will be a Point Circle if $|a|^2 = b$

PROBLEMS

Single Correct Answer Type

1. The number of solutions of the equation $z^2 + \bar{z} = 0$ is
(a) 1 (b) 2 (c) 3 (d) 4

2. Number of solutions of the equation $z^3 + \frac{9(\bar{z})^2}{|z|} = 0$, where z is a complex number is
(a) 2 (b) 3 (c) 6 (d) 5

3. If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\text{Im}(z_1 z_2) \neq 0$ then
(a) a, b, c are all real
(b) at least one of a, b, c is real
(c) at least one of a, b, c is imaginary
(d) all of a, b, c are imaginary

4. The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$ equals
(a) i (b) $i - 1$ (c) $-i$ (d) 0

5. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then
(a) $x = 2n + 1$, where n is any positive integer
(b) $x = 4n$, where n is any positive integer
(c) $x = 2n$, where n is any positive integer
(d) $x = 4n + 1$, where n is any positive integer.

6. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p} + \frac{y}{q}\right)}{(p^2 + q^2)}$ is equal to
(a) 2 (b) -1 (c) 1 (d) -2

7. Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is
(a) 48 (b) 32 (c) 40 (d) 80

8. Conjugate of a complex number is $\frac{1}{i-1}$, then the complex number
(a) $\frac{-1}{i+1}$ (b) $\frac{1}{i-1}$ (c) $\frac{-1}{i-1}$ (d) $\frac{1}{i+1}$

9. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for
(a) $x = n\pi$ (b) $x = 0$
(c) $x = (n + 1/2)\pi$ (d) no value of x

10. Given $z = (1 + i\sqrt{3})^{100}$, then $\frac{2\,\text{Re}(z)}{\sqrt{3}\,\text{Im}(z)}$ equals
(a) 2^{100} (b) 2^{50} (c) 2/3 (d) 3/2

11. For positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, $(i = \sqrt{-1})$, is a real number, if and only if
(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$

12. For a complex number z, the minimum value of $|z| + |z - e^{i\alpha}|$ is
(a) 0 (b) 1
(c) 2 (d) none of these

13. If $z = x + iy$ and $x^2 + y^2 = 16$, then the range of $||x| - |y||$ is
(a) [0, 4] (b) [0, 2] (c) [2, 4] (d) none of these

14. The number of complex number z satisfying $|z - 3 - i| = |z - 9 - i|$ and $|z - 3 + 3i| = 3$ are
(a) one (b) two (c) four (d) none of these

15. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is
(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

16. Let z and ω be two non zero complex numbers such that $|z| = |\omega|$ and $\arg z + \arg \omega = \pi$, then z equals
(a) ω (b) $-\omega$ (c) $\bar{\omega}$ (d) $-\bar{\omega}$

17. Let z and ω be two complex numbers such that $|z| \le 1, |\omega| \le 1, |z + i\omega| = |z - i\bar{\omega}| = 2$ then z is equal to
(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

Multiple Correct Answer Type

18. If z_1 and z_2 are two uni-modular complex numbers where $z_1 = a + ib$ and $z_2 = c + id$ and $\text{Re}(z_1 \cdot z_2) = 0$ then the pair of complex numbers $\omega_1 = a + ic$ and $\omega_2 = b + id$ satisfies :
(a) $|\omega_1| = 1$ (b) $|\omega_2| = 1$
(c) $\text{Re}(\omega_1\omega_2) = 0$ (d) none of these

19. If z is a complex number satisfying $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$ then z lies on
(a) pair of straight line
(b) circle
(c) parabola
(d) ellipse

20. If z_1 and z_2 are two complex number where $z_1 = 12 + 5i$ and $|z_2| = 4$ then
(a) maximum $(|z_1 + iz_2|) = 17$
(b) minimum $(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$

(c) minimum $\left|\dfrac{z_1}{z_2+\dfrac{4}{z_2}}\right| = \dfrac{13}{5}$

(d) maximum $\left|\dfrac{z_1}{z_2+\dfrac{4}{z_2}}\right| = \dfrac{13}{3}$

21. If z_1 and z_2 are two complex numbers in the argand plane and lies on two concentric circles $|z| = 1$ and $|z| = 2$ respectively, then

(a) $3 \le |z_1 - 2z_2| \le 5$ (b) $1 \le |z_1 + z_2| \le 3$
(c) $|z_1 - 3z_2| \ge 5$ (d) $|z_1 - z_2| \ge 1$

22. If the equations $\bar{a}z + a\bar{z} + b = 0$ and $\bar{a}z - a\bar{z} + b_1 = 0$ represent two lines L_1 and L_2 in the complex plane, then

(a) L_1 and L_2 are perpendicular
(b) b is purely real
(c) b_1 is purely imaginary
(d) b_1 is purely real

23. The complex number satisfying $|z+\bar{z}|+|z-\bar{z}|=2$ and $|z + i| + |z - i| = 2$ is/are

(a) i (b) $-i$ (c) $1 + i$ (d) $1 - i$

24. The value(s) of $(-8i)^{1/3}$ is/are

(a) $\sqrt{3}-i$ (b) $\sqrt{3}+i$
(c) $-\sqrt{3}-i$ (d) $-\sqrt{3}+i$

25. If $|(z - z_1)/(z - z_2)| = 7$, where z_1 and z_2 are fixed complex numbers and z is a variable complex number, then z lies on a

(a) circle with z_1 as its interior point
(b) circle with z_2 as its interior point
(c) circle with z_1 as its exterior point
(d) circle with z_2 as its exterior point

26. If z_r (where $r = 0, 1, 2, 3, \ldots\ldots, n - 1$) be the roots of equation $x^n - 1 = 0$ and ω be a non-real complex cube root of unity, then the product $\prod_{r=1}^{n-1}(\omega - z_r)$ can be equal to

(a) 0 (b) 1 (c) -1 (d) $1 + \omega$

27. Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $P(x)$ is divisible by $(x - 1)$ but $Q(x)$ is not divisible by $x - 1$
(b) $Q(x)$ is divisible by $(x - 1)$ but $P(x)$ is not divisible by $x - 1$
(c) both $P(x)$ and $Q(x)$ are divisible by $x - 1$
(d) $f(x)$ is divisible by $x - 1$

Comprehension Type

Paragraph for Q. No. 28 to 30

Let A, B, C be three sets of complex numbers as defined below:

$A = \{z : \text{Im } z \ge 1\}$
$B = \{z : |z - 2 - i| = 3\}$
$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$

28. The number of elements in the set $A \cap B \cap C$ is

(a) 0 (b) 1 (c) 2 (d) ∞

29. Let z be any point in $A \cap B \cap C$. Then $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

30. Let z be any point in $A \cap B \cap C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then $|z| - |\omega| + 3$ lies between

(a) -6 and 3 (b) -3 and 6
(c) -6 and 6 (d) -3 and 9

Integer Answer Type

31. If the complex number z is such that $|z - 1| \le 1$ and $|z - 2| = 1$ if r and R is the minimum and maximum value of $|z|^2$ then $r + R$ is

32. If $\left|\dfrac{z-25}{z-1}\right| = 5$, find the value of $|z|$.

33. If $z = \dfrac{\sqrt{3}-i}{2}$ and $(z^{95} + i^{67})^{94} = z^n$, then the sum of the digits of the smallest positive integral value of n is

34. If $|z - 3 + i| = |z| \sin\left(\dfrac{\pi}{4} - \arg z\right)$ and the eccentricity of locus of z is e, then value of $3e^2$ is

Matrix Match Type

35. Match the following.

Column-I		Column-II	
(i)	$z^4 - 16 = 0$	A.	$z = 2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)$
(ii)	$z^4 + 16 = 0$	B.	$z = 2\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)$
(iii)	$iz^4 + 16 = 0$	C.	$z = 2\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$
(iv)	$iz^4 - 16 = 0$	D.	$z = 2(\cos 0 + i \sin 0)$

SOLUTIONS

1. (d): Let $z = x + iy$, so that $\bar{z} = x - iy$

$\therefore\ z^2 + \bar{z} = 0 \Rightarrow (x^2 - y^2 + x) + i(2xy - y) = 0$

Equating real and imaginary parts, we get

$x^2 - y^2 + x = 0$ ...(i)

and $2xy - y = 0 \Rightarrow y = 0$ or $x = 1/2$

If $y = 0$, then (i) gives, $x^2 + x = 0$

$\Rightarrow x = 0$ or $x = -1$

If $x = 1/2$, then (i) gives,

$$y^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

Hence, there are four solutions in all.

2. (d): We have, $z^3 + \dfrac{9(\bar{z})^2}{|z|} = 0$

Let $z = re^{i\theta} \Rightarrow r^3e^{i3\theta} + 9re^{-i2\theta} = 0$

Since 'r' cannot be zero.

$\Rightarrow r^2e^{i5\theta} = -9$ which will hold for $r = 3$ and 5 distinct values of 'θ'.

Thus, there are five solutions.

3. (c): $\because\ az^2 + bz + c = 0$ ...(i)

and z_1, z_2 (roots of (i)) are such that $\text{Im}(z_1z_2) \neq 0$

$\Rightarrow$ z_1 and z_2 are not conjugates of each other.

$\Rightarrow$ Complex roots of (i) are not conjugate of each other

$\Rightarrow$ Coefficients a, b, c can not all be real.

$\Rightarrow$ Atleast one of a, b, c is imaginary.

4. (b): $\displaystyle\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13} i^n(1+i) = (1+i)\sum_{n=1}^{13} i^n$

$= (1 + i)(i + i^2 + i^3 + \ldots + i^{13})$

$= (1+i)\left\{\dfrac{i(1-i^{13})}{1-i}\right\} = (1 + i)i = -1 + i$

5. (b): Given, $\left(\dfrac{1+i}{1-i}\right)^x = 1$

$\Rightarrow \left\{\dfrac{(1+i)(1+i)}{(1-i)(1+i)}\right\}^x = 1 \Rightarrow \left(\dfrac{1+i^2+2i}{1-i^2}\right)^x = 1$

$\Rightarrow \left(\dfrac{2i}{2}\right)^x = 1 \Rightarrow i^x = 1$

$\Rightarrow$ x = an integral multiple of 4. [$\because\ i^4 = 1$]

$\therefore$ $x = 4n$, where n is an integer.

6. (d): Given, $z^{1/3} = p + iq$

$\Rightarrow z = (p + iq)^3 \Rightarrow x - iy = p^3 - 3pq^2 + i(3p^2q - q^3)$

Equating real and imaginary parts, we get

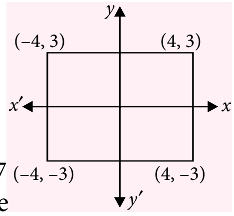
$x = p^3 - 3pq^2 \Rightarrow \dfrac{x}{p} = p^2 - 3q^2$ ...(i)

and $-y = 3p^2q - q^3 \Rightarrow \dfrac{y}{q} = q^2 - 3p^2$ ...(ii)

Adding (i) and (ii), we get

$$\frac{x}{p} + \frac{y}{q} = -2(p^2 + q^2)$$

7. (a): $z\bar{z}^3 + \bar{z}z^3 = 350$



$\Rightarrow z\bar{z}(z^2 + \bar{z}^2) = 350$

$\Rightarrow z\bar{z}\{(z+\bar{z})^2 - 2z\bar{z}\} = 350$

$\Rightarrow (x^2 + y^2)(4x^2 - 2x^2 - 2y^2) = 350$

$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 175 = 25 \times 7$

As x and y are integers, so we have

$x^2 + y^2 = 25$ and $x^2 - y^2 = 7$

$\Rightarrow (x, y) = (4, 3), (-4, 3), (-4, -3)$ and $(4, -3)$

$\therefore$ Area of rectangle $= 8 \times 6 = 48$

8. (a): We have, $\bar{z} = \dfrac{1}{i-1}$

We have $z = \overline{(\bar{z})}$ giving $z = \dfrac{1}{\overline{(i-1)}} = \dfrac{1}{-i-1} = \dfrac{-1}{i+1}$

9. (d): As $\overline{(\sin x + i\cos 2x)} = \cos x - i\sin 2x$

$\Rightarrow$ $\sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow$ $\tan x = 1$ and $\tan 2x = 1$

Since, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$. Therefore, $\tan x \neq \pm 1$

Hence, there is no value of x satisfying $\tan x = 1$ and $\tan 2x = 1$ simultaneously.

10. (c): $z = (1+i\sqrt{3})^{100} = 2^{100}\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^{100}$

$= 2^{100}\left(\cos\dfrac{100\pi}{3} + i\sin\dfrac{100\pi}{3}\right)$

$= 2^{100}\left(-\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right) = 2^{100}\left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)$

Now, $\dfrac{2\,\text{Re}(z)}{\sqrt{3}\,\text{Im}(z)} = \dfrac{2\times 2^{100}\cdot(-1/2)}{\sqrt{3}\cdot 2^{100}\cdot(-\sqrt{3}/2)} = \dfrac{2}{3}$

11. (d): $\{(1+i)^{n_1} + (1-i)^{n_1}\} + \{(1+i)^{n_2} + (1-i)^{n_2}\}$ is a real number for all n_1 and $n_2 \in R$.

{as, $z + \bar{z} = 2\,\text{Re}(z) \Rightarrow (1+i)^n + (1-i)^n$ is a real number for all $n \in R$}

12. (b): We are finding out the sum of distances of a complex number z from origin and $(\cos\alpha, \sin\alpha)$. This sum will be minimum if z lies on the line joining the two points. So, minimum value of sum will be the distance between the points $(0, 0)$ and $(\cos\alpha, \sin\alpha)$ *i.e.*, 1.

13. (a): Here $x = 4\cos\theta$, $y = 4\sin\theta$.

$\therefore\ ||x| - |y|| = ||4\cos\theta| - |4\sin\theta||$

$= 4||\cos\theta| - |\sin\theta|| = 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$

$= 4\sqrt{1 - |\sin 2\theta|}$

Hence, the range is $[0, 4]$.

14. (a) : Let $z = x + iy$. Then, $|z - 3 - i| = |z - 9 - i|$

$\Rightarrow \sqrt{(x-3)^2+(y-1)^2} = \sqrt{(x-9)^2+(y-1)^2} \Rightarrow x = 6$

Also, $|z - 3 + 3i| = 3$

$\Rightarrow \sqrt{(x-3)^2+(y+3)^2} = 3$

For $x = 6$, $y = -3$. $\quad \therefore \; z = 6 - 3i$

15. (a) : $|z_1| = |z_2| = |z_3| = 1$

Now, $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

Similarly, $z_2\bar{z}_2 = 1$, $z_3\bar{z}_3 = 1$

Now, $\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right| = 1$

$\Rightarrow \left|\dfrac{z_1\bar{z}_1}{z_1}+\dfrac{z_2\bar{z}_2}{z_2}+\dfrac{z_3\bar{z}_3}{z_3}\right| = |\bar{z}_1+\bar{z}_2+\bar{z}_3| = 1$

$\Rightarrow |z_1 + z_2 + z_3| = 1$

16. (d) : We have to find z in terms of ω under given conditions.

Let $\omega = re^{i\theta}$ $\therefore$ $\bar{\omega} = re^{-i\theta}$

$\Rightarrow z = re^{i(\pi - \theta)} = re^{i\pi} \cdot e^{-i\theta} = -re^{-i\theta} = -\bar{\omega}$

17. (c) : We have $|z + i\omega| = |z - i\bar{\omega}| = 2$

$\Rightarrow |z - (-i\omega)| = |z - (-i\bar{\omega})| = 2$

$\Rightarrow |z - (-i\omega)| = |z - (-i\bar{\omega})|$

$\therefore$ z lies on the perpendicular bisector of the line joining $-i\omega$ and $-i\bar{\omega}$. Since $-i\bar{\omega}$ is the mirror image of $-i\omega$ in the x-axis, the locus of z is the x-axis.

Let $z = x + iy$ and $y = 0$

Now $|z| \le 1 \Rightarrow x^2 + 0^2 \le 1 \Rightarrow -1 \le x \le 1$.

$\therefore$ z may take values given in option (c).

18. (a, b, c) : $z_1 = a + ib$ or $z_1 = \cos\theta + i\sin\theta$

$z_2 = c + id$ or $z_2 = \cos\alpha + i\sin\alpha$

$z_1z_2 = \cos(\theta + \alpha) + i\sin(\theta + \alpha)$

$\therefore$ $\text{Re}(z_1z_2) = \cos(\theta + \alpha) = 0$

$\Rightarrow \theta + \alpha = \dfrac{\pi}{2}$

$\omega_1 = \cos\theta + i\cos\alpha = \cos\theta + i\cos\left(\dfrac{\pi}{2} - \theta\right)$

$= \cos\theta + i\sin\theta = e^{i\theta}$

$\Rightarrow |\omega_1| = 1$

$\omega_2 = \sin\theta + i\sin\alpha = \sin\left(\dfrac{\pi}{2} - \alpha\right) + i\sin\alpha$

$= \cos\alpha + i\sin\alpha = e^{i\alpha}$

$\Rightarrow |\omega_2| = 1$

$\therefore$ $\omega_1\omega_2 = e^{i(\theta + \alpha)}$

$\Rightarrow \text{Re}(\omega_1\omega_2) = \cos(\theta + \alpha) = 0 \quad \left[\because \alpha + \theta = \dfrac{\pi}{2}\right]$

19. (a) : $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$

Let $z = x + iy$, then

$|x + iy - ix| = |x + iy - y|$

$\Rightarrow x^2 + (y - x)^2 = (x - y)^2 + y^2 \Rightarrow x^2 = y^2$ *i.e.* $y = \pm x$

20. (a, b, c, d) : $z_1 = 12 + 5i$, $|z_2| = 4$

$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\min(|z_1 + (1 + i)z_2|) = ||z_1| - |1 + i||z_2|| = 13 - 4\sqrt{2}$

$\left|z_2 + \dfrac{4}{z_2}\right| \le |z_2| + \dfrac{4}{|z_2|} = 4 + 1 = 5$

$\left|z_2 + \dfrac{4}{z_2}\right| \ge |z_2| - \dfrac{4}{|z_2|} = 4 - 1 = 3$

$\Rightarrow 3 \le \left|z_2 + \dfrac{4}{z_2}\right| \le 5 \Rightarrow \dfrac{1}{3} \ge \dfrac{1}{\left|z_2 + \dfrac{4}{z_2}\right|} \ge \dfrac{1}{5}$

$\Rightarrow \dfrac{13}{3} \ge \dfrac{|z_1|}{\left|z_2 + \dfrac{4}{z_2}\right|} \ge \dfrac{13}{5}$

21. (a, b, c, d) : $|z_1| = 1$, $|z_2| = 2$

(a) $||z_1| - 2| - z_2|| \le |z_1 - 2z_2| \le |z_1| + 2| - z_2|$

$\Rightarrow |1 - 2(2)| \le |z_1 - 2z_2| \le 1 + 2(2)$

$\Rightarrow 3 \le |z_1 - 2z_2| \le 5$

(b) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

$\Rightarrow 1 \le |z_1 + z_2| \le 3$

(c) $||z_1| - 3|z_2|| \le |z_1 - 3z_2| \le |z_1| + 3| - z_2|$

$\Rightarrow 5 \le |z_1 - 3z_2| \le 7$

(d) $||z_1| - |z_2|| \le |z_1 - z_2| \le |z| + |z_2|$

$\Rightarrow 1 \le |z_1 - z_2| \le 3$

22. (a, b, c) : Equation of a line perpendicular to the line $\bar{a}z + a\bar{z} + b = 0$ is $\bar{a}z - a\bar{z} + \lambda i = 0$ where λ is purely real.

23. (a, b) : Given, $|z + \bar{z}| + |z - \bar{z}| = 2$

$\Rightarrow |2\text{Re}(z)| + |2\text{Im}(z)| = 2 \Rightarrow |x| + |y| = 1$

Which is the locus of a square.

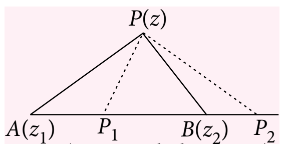
Also, $|z + i| + |z - i| = 2$ represents a line. *i.e.* $x = 0$

Hence, options (a), (b) are the correct answers.

24. (a, c) : $(-8i)^{1/3} = (8i^3)^{1/3} = 2i, 2i\omega, 2i\omega^2$ where $\omega = \dfrac{-1+\sqrt{3}i}{2}$

Hence, roots are $2i$, $-\sqrt{3} - i$, $\sqrt{3} - i$

25. (b, c) : Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2 respectively.



Hence, $AP_1 : P_1B \equiv PA : PB \equiv 7 : 1$(internal division)

$AP_2 : P_2B \equiv PA : PB \equiv 7 : 1$ (external division)

Thus P_1 and P_2 are fixed points. Also, $\angle P_1PP_2 = \dfrac{\pi}{2}$

Thus P lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside and $A(z_1)$ lies outside this circle.

26. (a, b, d) : $x^n - 1 = (x-1)(x-z_1)(x-z_2)\dots(x-z_{n-1})$

$\Rightarrow \dfrac{x^n-1}{x-1} = (x-z_1)(x-z_2)\dots(x-z_{n-1})$

Putting $x = \omega$, we have

$$\prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0, & \text{if } n = 3k, k \in Z \\ 1, & \text{if } n = 3k+1, k \in Z \\ 1+\omega, & \text{if } n = 3k+2, k \in Z \end{cases}$$

27. (c, d) : We have, $x^2 + x + 1 = (x-\omega)(x-\omega^2)$

Since $f(x)$ is divisible by $x^2 + x + 1$

$\therefore \quad f(\omega) = 0, f(\omega^2) = 0,$

$\therefore \quad P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0 \quad$ ...(i)

and $P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0$ ...(ii)

Solving (i) and (ii), we obtain

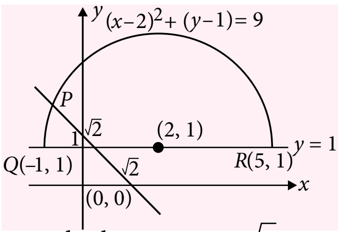
$P(1) = 0$ and $Q(1) = 0$

Therefore, both $P(x)$ and $Q(x)$ are divisible by $x - 1$. Hence, $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$ and so by $x - 1$. Since $f(x) = P(x^3) + xQ(x^3)$, we get $f(x)$ is divisible by $x - 1$.

28. (b) : A is the set of points $(x, y,), y \geq 1$

B is the set of points on the circle $(x-2)^2 + (y-1)^2 = 3^2$

Also, $\mathrm{Re}(1-i)(x+iy) = \sqrt{2}$



$\therefore \quad C$ is the set of points on the line $x + y = \sqrt{2}$.

$A \cap B \cap C$ contains the single point P which is the point of intersection of the line $x + y = \sqrt{2}$ and the circle.

29. (c) : Since z is any point P on the circle then

$|z + 1 - i|^2 + |z - 5 - i|^2 = PQ^2 + PR^2$

where $Q = (-1, 1)$, $R = (5, 1)$

Hence, $PQ^2 + PR^2 = QR^2 = 36$

30. (d): $|\omega - 2 - i| < 3 \Rightarrow \omega$ is a point inside the circle.

But P is a point on the circle.

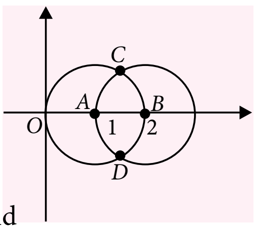
$\therefore \quad |z - \omega| < 6$ (= diameter)

$||z| - |\omega|| < |z - \omega| \Rightarrow ||z| - |\omega|| < 6$

$\therefore \quad -6 < |z| - |\omega| < 6 \Rightarrow -3 < |z| - |\omega| + 3 < 9$

31. (4): $|z-1| \leq 1$ representsthe interior and boundary of the circle with centre at $1 + 0i$ and radius 1 and $|z - 2| = 1$ represents circle with centre at $2 + 0i$ and radius 1.

Clearly the points z satisfying $|z - 1| \leq 1$ and $|z - 2| = 1$ lie on the arc DAC.



$\therefore \quad OA \leq |z| \leq OC$ (= OD)

As $\angle OCB = \pi/2$, $OC^2 = OB^2 - BC^2 = 4 - 1 = 3 \therefore \; OC = \sqrt{3}$

Thus, $R = |z_{max}|^2 = OC^2 = 3$ and

$r = |z_{min}|^2 = OA^2 = 1$

Hence, $r + R = 4$

32. (5): $\left|\dfrac{z-25}{z-1}\right| = 5 \Rightarrow |z-25|^2 = 25|z-1|^2$

$\Rightarrow \quad |z|^2 - 25z - 25\bar{z} + 625 = 25\{|z|^2 - z - \bar{z} + 1\}$

$\Rightarrow \quad |z| = 5$

33. (1): We have, $z = \dfrac{\sqrt{3}-i}{2} = -i\left(\dfrac{1+i\sqrt{3}}{2}\right) = i\omega^2$

where ω is a cube root of unity.

Now, $z^{95} = i^{95}\omega^{190} = (i^{4 \times 23} \times i^3)(\omega^{3 \times 63} \times \omega^1)$

$= (-i)(\omega) \qquad [\because \; i^4 = 1 \text{ and } \omega^3 = 1]$

and $i^{67} = i^{4 \times 16} \times i^3 = -i$

$\therefore \quad (z^{95} + i^{67})^{94} = [-i(1+\omega)]^{94} = (i\omega^2)^{94}$

$[\because \; 1 + \omega + \omega^2 = 0]$

$= (i^{4 \times 23} \times i^2)(\omega^{3 \times 62} \times \omega^2) = i^2 \times \omega^2$

According to the given condition, we have

$z^n = (z^{95} + i^{67})^{94}$

$\Rightarrow \quad (i\omega^2)^n = i^2\omega^2 \Rightarrow i^{n-2} \times \omega^{2n-2} = 1$

Then, $n - 2 = 4a$ and $2n - 2 = 3b$ where $a, b \in I$

Eliminating n, we have

$2(4a + 2) - 2 = 3b \quad$ *i.e.*, $b = \dfrac{8a+2}{3}$

Smallest integral values for a, b are $a = 2$ and $b = 6$.

and hence, smallest integral value of n is

$n = 4 \times 2 + 2 = 10$

34. (3): Given, $|z - 3 + i| = |z|\sin\left(\dfrac{\pi}{4} - \arg z\right)$

$\Rightarrow \quad |(x-3) + i(y+1)| = |z|\left(\dfrac{1}{\sqrt{2}}\cos\theta - \dfrac{1}{\sqrt{2}}\sin\theta\right)$

(where $\theta = \arg z$)

$\Rightarrow \quad \sqrt{(x-3)^2 + (y+1)^2} = \dfrac{1}{\sqrt{2}}|z|\cos\theta - \dfrac{1}{\sqrt{2}}|z|\sin\theta$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y+1)^2} = \dfrac{1}{\sqrt{2}}(x-y)$

Which is a parabola with focus $(3, -1)$ and directrix $x - y = 0$.

So, its eccentricity $e = 1$

35. (i) – (D); (ii) – (C); (iii) – (A); (iv) – (B)

(i) $z^4 - 16 = 0 \Rightarrow z^4 = 16 = 16(\cos 0 + i \sin 0)$

$\Rightarrow \quad z = 2(\cos 0 + i \sin 0)^{1/4} = 2(\cos 0 + i \sin 0)$

(ii) $z^4 + 16 = 0 \Rightarrow z^4 = -16 = 16(\cos\pi + i\sin\pi)$

$\Rightarrow \quad z = 2(\cos\pi + i\sin\pi)^{1/4} = 2\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

(iii) $iz^4 + 16 = 0 \Rightarrow z^4 = 16i = 16\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$

$\Rightarrow \quad z = 2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)$

(iv) $iz^4 - 16 = 0 \Rightarrow z^4 = -16i = 16\left(\cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}\right)$

$\Rightarrow \quad z = 2\left(\cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}\right)^{1/4} = 2\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)$

CLASS XI Series 3

ACE YOUR WAY CBSE

Trigonometric Functions and Principle of Mathematical Induction

HIGHLIGHTS

TRIGONOMETRIC FUNCTIONS

Measure of an Angle : The measure of an angle is the amount of rotation of a ray about a fixed point from its initial position to the terminal position.

UNITS OF MEASUREMENT OF ANGLES

SEXAGESIMAL SYSTEM

In this system,

1 right angle = 90° (90 degrees)

1° *i.e.*, 1 degree = 60′ (60 minutes)

1′ *i.e.*, 1 minute = 60″ (60 seconds)

CENTESIMAL SYSTEM

In this system,

1 right angle = 100^g (100 grades)

1^g *i.e.*, 1 grade = 100′ (100 minutes)

1′ *i.e.*, 1 minute = 100″ (100 seconds)

CIRCULAR SYSTEM

In this system,

1 right angle $=\left(\frac{\pi}{2}\right)^c$ or $\frac{\pi}{2}$ radians

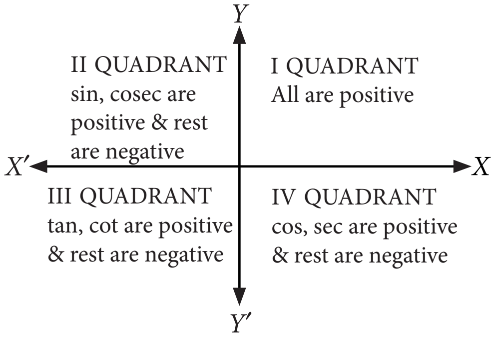
1° *i.e.*, 1 degree $=\frac{\pi}{180}$ radian

Note : If in a circle of radius r, an arc of length l, subtends an angle θ radians at the centre, then $\theta=\frac{l}{r}$

CONVERSION OF DEGREE MEASURE TO RADIAN MEASURE AND VICE-VERSA

- Radian measure $=\frac{\pi}{180}\times$ Degree measure
- Degree measure $=\frac{180}{\pi}\times$ Radian measure

SIGN CONVENTION OF TRIGONOMETRIC FUNCTIONS



DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

$y = \sin x$
Domain $= R$
Range $= [-1, 1]$

$y = \cos x$
Domain $= R$
Range $= [-1, 1]$

$y = \tan x$
$\text{Domain} = R - \left\{(2n-1)\frac{\pi}{2}, n \in Z\right\}$
Range $= R$

$y = \operatorname{cosec} x$
Domain $= R - \{n\pi, n \in Z\}$
Range $= R - (-1, 1)$

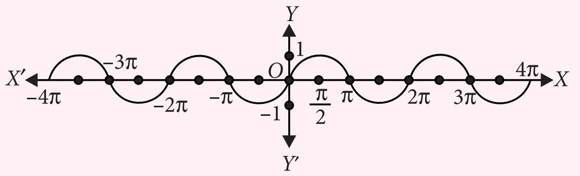
$y = \sec x$
$\text{Domain} = R - \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$
Range $= R - (-1, 1)$

$y = \cot x$
Domain $= R - \{n\pi, n \in Z\}$
Range $= R$

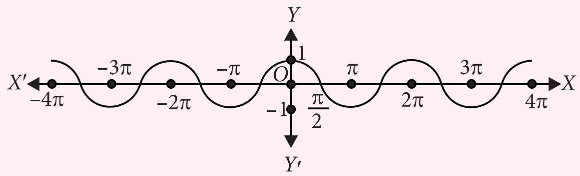
VALUES OF T-FUNCTIONS FOR SOME PARTICULAR ANGLES

T-FUNCTIONS / ANGLES	sin	cos	tan	cot	sec	cosec
0	0	1	0	not defined	1	not defined
$\frac{\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{2}(\sqrt{3}-1)$	$\sqrt{2}(\sqrt{3}+1)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$\frac{\pi}{2}$	1	0	not defined	0	not defined	1
π	0	-1	0	not defined	-1	not defined
$\frac{3\pi}{2}$	-1	0	not defined	0	not defined	-1
2π	0	1	0	not defined	1	not defined

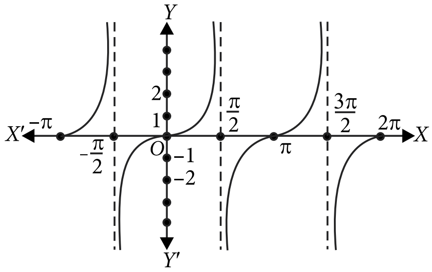
GRAPHS OF T-FUNCTIONS



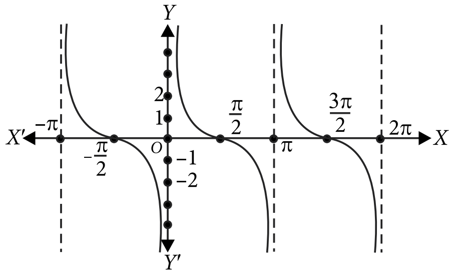
$y = \sin x$



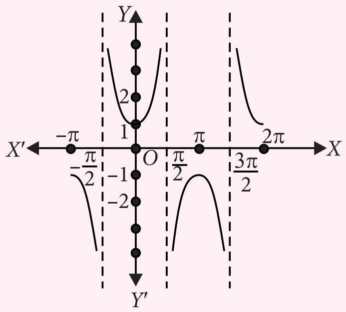
$y = \cos x$



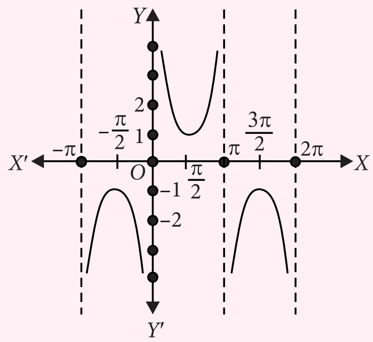
$y = \tan x$



$y = \cot x$



$y = \sec x$



$y = \operatorname{cosec} x$

TRIGONOMETRIC FUNCTIONS OF SOME ALLIED ANGLES IN TERMS OF θ

T-Ratios / Allied Angles	sin	cos	tan	cot	sec	cosec
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$	$-\cot\theta$	$\sec\theta$	$-\operatorname{cosec}\theta$
$\left(\frac{\pi}{2} \pm \theta\right)$	$\cos\theta$	$\mp\sin\theta$	$\mp\cot\theta$	$\mp\tan\theta$	$\mp\operatorname{cosec}\theta$	$\sec\theta$
$(\pi \pm \theta)$	$\mp\sin\theta$	$-\cos\theta$	$\pm\tan\theta$	$\pm\cot\theta$	$-\sec\theta$	$\mp\operatorname{cosec}\theta$
$\left(\frac{3\pi}{2} \pm \theta\right)$	$-\cos\theta$	$\pm\sin\theta$	$\mp\cot\theta$	$\mp\tan\theta$	$\pm\operatorname{cosec}$	$-\sec\theta$
$(2\pi \pm \theta)$	$\pm\sin\theta$	$\cos\theta$	$\pm\tan\theta$	$\pm\cot\theta$	$\sec\theta$	$\pm\operatorname{cosec}\theta$

IMPORTANT FORMULAE

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
- $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$
- $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$
- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

  $= 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
- $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
- $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\cos A - \cos B = 2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B-A}{2}\right)$
- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$
- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
- $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}\begin{bmatrix} +\text{ ve, if } \frac{A}{2} \text{ lies in I or II quadrants} \\ -\text{ve, if } \frac{A}{2} \text{ lies in III or IV quadrants}\end{bmatrix}$
- $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}\begin{bmatrix} +\text{ve, if } \frac{A}{2} \text{ lies in I or IV quadrants} \\ -\text{ve, if } \frac{A}{2} \text{ lies in II or III quadrants}\end{bmatrix}$
- $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}\begin{bmatrix} +\text{ve, if } \frac{A}{2} \text{ lies in I or III quadrants} \\ -\text{ve, if } \frac{A}{2} \text{ lies in II or IV quadrants}\end{bmatrix}$

TRIGONOMETRIC EQUATIONS

Equations involving one or more trigonometric functions of unknown angle are called trigonometric equations.

Principal Solution
Solutions of trigonometric equation lying in the interval $[0, 2\pi)$

General Solution
All possible solutions of a trigonometric equation.

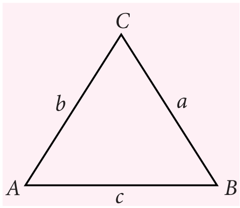
SOLUTION OF TRIGONOMETRIC EQUATIONS

GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS

- $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$
- $\cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2},\ n \in Z$
- $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in Z$
- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ n \in Z$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in Z$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in Z$
- $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
- $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$
- $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha,\ n \in Z$

APPLICATIONS OF SINE AND COSINE FORMULAE

Let A, B and C be the angles of a triangle and a, b and c be the lengths of their opposite sides respectively. Then,



1. **Sine Rule**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. **Cosine Rule**

$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = c^2 + a^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

3. **Napier's Analogy**

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$$
$$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$$
$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$$

Note : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

PRINCIPLE OF MATHEMATICAL INDUCTION

First principle of mathematical induction

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true *i.e.* $P(n)$ is true for $n = 1$, and

(ii) $P(m + 1)$ is true, whenever $P(m)$ is true.
i.e. $P(m)$ is true $\Rightarrow P(m + 1)$ is true.

Then, $P(n)$ is true for all natural numbers n.

Second principle of mathematical induction

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true *i.e.* $P(n)$ is true for $n = 1$, and

(ii) $P(m + 1)$ is true, whenever $P(n)$ is true for all n, where $1 \le n \le m$.

Then, $P(n)$ is true for all natural numbers.

PROBLEMS

Very Short Answer Type

1. Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.
2. If $3 \sin \theta + 5\cos\theta = 5$, show that $5 \sin \theta - 3 \cos \theta = \pm 3$
3. Find the general solution of the equation : $4\sin^2 x = 1$
4. Prove that : $\cos(45° - A) \cos (45° - B) - \sin(45° - A) \sin (45° - B) = \sin (A + B)$
5. Solve the equation : $\cot x + \tan x = 2 \operatorname{cosec} x$.

Long Answer Type - I

6. Find the general solution of equation $\sin 2x + \sin 4x + \sin 6x = 0$.
7. If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7", and if $P(r)$ is true, prove that $P(r + 1)$ is true.
8. If $T_n = \sin^n \theta + \cos^n\theta$, prove that $\dfrac{T_3 - T_5}{T_1} = \dfrac{T_5 - T_7}{T_3}$
9. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, then prove that one of the values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.
10. Show that $\cos^2 A + \cos^2 B - 2\cos A\cos B \cdot \cos(A + B) = \sin^2(A + B)$

Long Answer Type - II

11. In any ΔABC, prove that $a(\cos C - \cos B) = 2(b - c)\cos^2\dfrac{A}{2}$
12. Prove that $\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma)$ $= 4\cos\dfrac{\alpha+\beta}{2}\cdot\cos\dfrac{\beta+\gamma}{2}\cdot\cos\dfrac{\gamma+\alpha}{2}$
13. Using the principle of mathematical induction prove that
$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1} \text{ for all } n \in N.$$
14. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, prove that
$$\tan\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$$
15. An object is observed from three points A, B, C in the same horizontal line passing through the base of the object. The angle of elevation at B is twice and at C thrice that at A. If $AB = a$, $BC = b$, then prove that the height of the object is $\dfrac{a}{2b}\sqrt{(a+b)(3b-a)}$.

SOLUTIONS

1. Here, $r = 25$ cm and $l = 11$ cm
Let the measure of the required angle be θ.
Then, $\theta = \left(\dfrac{l}{r}\right) = \left(\dfrac{11}{25}\right)^c = \left(\dfrac{11}{25}\times\dfrac{180}{\pi}\right)^\circ$
$= \left(\dfrac{11}{25}\times\dfrac{7}{22}\times 180\right)^\circ = \left(\dfrac{126}{5}\right)^\circ = 25°12'$

2. Given, $3 \sin\theta + 5 \cos\theta = 5$ ...(i)
Let $5 \sin \theta - 3 \cos \theta = x$ ...(ii)
Squaring (i) and (ii), and adding, we get
$(9 \sin^2 \theta + 25 \cos^2 \theta + 30 \sin \theta \cos \theta)$
$+ (25 \sin^2 \theta + 9\cos^2 \theta - 30 \sin \theta \cos \theta) = 25 + x^2$
$\Rightarrow 9(\sin^2 \theta + \cos^2 \theta) + 25 (\sin^2 \theta + \cos^2 \theta) = 25 + x^2$
$\Rightarrow 34 = 25 + x^2 \Rightarrow x^2 = 9, \;\therefore\; x = \pm 3$

3. $4\sin^2 x = 1 \Rightarrow \sin^2 x = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2 = \sin^2\dfrac{\pi}{6}$
$\Rightarrow \sin^2 x = \sin^2\dfrac{\pi}{6} \Rightarrow x = n\pi \pm \dfrac{\pi}{6}$, where $n \in Z$.

4. L.H.S. $= \cos(45° - A) \cos (45° - B) - \sin(45° - A) \sin (45° - B)$
$= \cos \{(45° - A) + (45° - B)\} = \cos \{90° - (A + B)\}$
$= \sin(A + B) =$ R.H.S.

5. We have, $\cot x + \tan x = 2 \operatorname{cosec} x$
$\Rightarrow \dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} = \dfrac{2}{\sin x}$
$\Rightarrow \cos^2 x + \sin^2 x = 2\cos x \Rightarrow 1 = 2\cos x \Rightarrow \cos x = \dfrac{1}{2}$
$\Rightarrow \cos x = \cos\dfrac{\pi}{3} \Rightarrow x = 2n\pi \pm \dfrac{\pi}{3}, \; n \in Z$

6. We have, $(\sin 6x + \sin 2x) + \sin 4x = 0$
$\Rightarrow 2\sin\left(\dfrac{6x+2x}{2}\right)\cos\left(\dfrac{6x-2x}{2}\right) + \sin 4x = 0$

$\Rightarrow$ $2\sin 4x \cos 2x + \sin 4x = 0$

$\Rightarrow$ $\sin 4x(2\cos 2x + 1) = 0$

$\Rightarrow$ $\sin 4x = 0$ or $2\cos 2x + 1 = 0$

$\Rightarrow$ $\sin 4x = 0$ or $\cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow$ $4x = n\pi$ or $2x = \left(2m\pi \pm \frac{2\pi}{3}\right)$, where $m, n \in Z$

$\Rightarrow$ $x = \frac{n\pi}{4}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in Z$

7. Given $P(r)$ is true. Therefore, $2^{3r} - 1$ is an integral multiple of 7.

We need to prove that $P(r+1)$ is true *i.e.* $2^{3(r+1)} - 1$ is an integral multiple of 7.

$\because$ $P(r)$ is true *i.e.* $2^{3r} - 1$ is an integral multiple of 7.

$\Rightarrow$ $2^{3r} - 1 = 7\lambda$, for some $\lambda \in N$

$\Rightarrow$ $2^{3r} = 7\lambda + 1$ ...(i)

Consider, $2^{3(r+1)} - 1 = 2^{3r} \times 2^3 - 1 = (7\lambda + 1) \times 8 - 1$ [Using (i)]

$\Rightarrow$ $2^{3(r+1)} - 1 = 56\lambda + 8 - 1 = 56\lambda + 7 = 7(8\lambda + 1)$

$\Rightarrow$ $2^{3(r+1)} - 1 = 7\mu$, where $\mu = 8\lambda + 1 \in N$

$\Rightarrow$ $2^{3(r+1)} - 1$ is an integral multiple of 7

$\Rightarrow$ $P(r+1)$ is true

8. $T_3 - T_5 = (\sin^3\theta + \cos^3\theta) - (\sin^5\theta + \cos^5\theta)$

$= \sin^3\theta(1 - \sin^2\theta) + \cos^3\theta(1 - \cos^2\theta)$

$= \sin^3\theta \cdot \cos^2\theta + \cos^3\theta \cdot \sin^2\theta$

$= \sin^2\theta \cdot \cos^2\theta(\sin\theta + \cos\theta)$

$\therefore$ $\frac{T_3 - T_5}{T_1} = \frac{\sin^2\theta\cos^2\theta(\sin\theta + \cos\theta)}{(\sin\theta + \cos\theta)}$

$= \sin^2\theta \cdot \cos^2\theta$ ...(i)

Again, $T_5 - T_7 = (\sin^5\theta + \cos^5\theta) - (\sin^7\theta + \cos^7\theta)$

$= \sin^5\theta(1 - \sin^2\theta) + \cos^5\theta(1 - \cos^2\theta)$

$= \sin^5\theta \cdot \cos^2\theta + \cos^5\theta \cdot \sin^2\theta$

$= \sin^2\theta \cdot \cos^2\theta(\sin^3\theta + \cos^3\theta)$

$\therefore$ $\frac{T_5 - T_7}{T_3} = \frac{\sin^2\theta \cdot \cos^2\theta(\sin^3\theta + \cos^3\theta)}{(\sin^3\theta + \cos^3\theta)}$

$= \sin^2\theta \cdot \cos^2\theta$ ...(ii)

$\therefore$ From (i) and (ii), $\frac{T_3 - T_5}{T_1} = \frac{T_5 - T_7}{T_3}$

9. $\tan^2\frac{\theta}{2} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{1 - \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}}{1 + \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}}$

$= \frac{1 - \cos\alpha\cos\beta - \cos\alpha + \cos\beta}{1 - \cos\alpha\cos\beta + \cos\alpha - \cos\beta}$

$= \frac{(1 - \cos\alpha) + \cos\beta(1 - \cos\alpha)}{(1 + \cos\alpha) - \cos\beta(1 + \cos\alpha)}$

$= \frac{(1 - \cos\alpha)(1 + \cos\beta)}{(1 + \cos\alpha)(1 - \cos\beta)} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2}$

$\therefore$ $\tan\frac{\theta}{2} = \pm\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$

Hence one of the values of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

10. L.H.S. $= \cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A + B)$

$= \cos^2 A + (1 - \sin^2 B) - 2\cos A\cos B\cos(A + B)$

$= 1 + (\cos^2 A - \sin^2 B) - 2\cos A\cos B\cos(A + B)$

$= 1 + \cos(A + B)\cdot\cos(A - B) - 2\cos A\cos B\cos(A + B)$

$= 1 + \cos(A + B)[\cos(A - B) - 2\cos A\cos B]$

$= 1 + \cos(A + B)[\cos A\cdot\cos B + \sin A\cdot\sin B - 2\cos A\cdot\cos B]$

$= 1 + \cos(A + B)[\sin A\cdot\sin B - \cos A\cdot\cos B]$

$= 1 - \cos(A + B)[\cos A\cdot\cos B - \sin A\cdot\sin B]$

$= 1 - \cos^2(A + B) = \sin^2(A + B) =$ R.H.S.

11. L.H.S. $= a(\cos C - \cos B)$

$= a\left[\frac{(a^2 + b^2 - c^2)}{2ab} - \frac{(a^2 + c^2 - b^2)}{2ac}\right]$

$= \frac{(a^2c + b^2c - c^3 - a^2b - bc^2 + b^3)}{2bc}$

$= \frac{(b^3 - c^3) + (b^2c - bc^2) - (a^2b - a^2c)}{2bc}$

$= \frac{(b^3 - c^3) + bc(b - c) - a^2(b - c)}{2bc}$

$= (b - c)\frac{[(b^2 + c^2 + bc) + bc - a^2]}{2bc}$

$= (b - c)\left\{\frac{(b^2 + c^2 - a^2)}{2bc} + \frac{2bc}{2bc}\right\}$

$= (b - c)(1 + \cos A) = 2(b - c)\cos^2\frac{A}{2} =$ R.H.S.

12. L.H.S. $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)$

$= (\cos\alpha + \cos\beta) + [\cos(\alpha + \beta + \gamma) + \cos\gamma]$

$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + 2\cos\left(\frac{\alpha + \beta + \gamma + \gamma}{2}\right) \cdot \cos\left(\frac{\alpha + \beta + \gamma - \gamma}{2}\right)$

$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + 2\cos\left(\frac{\alpha + \beta + 2\gamma}{2}\right) \cdot \cos\left(\frac{\alpha + \beta}{2}\right)$

$= 2\cos\left(\frac{\alpha + \beta}{2}\right)\left[\cos\left(\frac{\alpha - \beta}{2}\right) + \cos\left(\frac{\alpha + \beta + 2\gamma}{2}\right)\right]$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cdot\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right]$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[2\cos\left(\frac{\alpha+\gamma}{2}\right)\cdot\cos\left(\frac{\beta+\gamma}{2}\right)\right]$$

$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{R.H.S.}$$

13. Let $P(n)$ be the statement given by

$$P(n): 1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+.....+n} = \frac{2n}{n+1}$$

We have, $P(1): 1 = \frac{2\times 1}{1+1}$

Clearly, $\frac{2\times 1}{1+1} = \frac{2}{2} = 1 \quad \therefore \quad 1 = \frac{2\times 1}{1+1}$

So, $P(1)$ is true.

Let $P(m)$ be true. Then,

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+.....+m} = \frac{2m}{m+1} \quad ...(i)$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+.....+m} + \frac{1}{1+2+3+.....+(m+1)} = \frac{2(m+1)}{(m+1)+1}$$

$$\therefore \quad \text{L.H.S.} = 1+\frac{1}{1+2}+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+....+m} + \frac{1}{1+2+3+.....+(m+1)}$$

$$= \frac{2m}{m+1}+\frac{1}{1+2+3+.....+(m+1)} \quad \text{[Using (i)]}$$

$$= \frac{2m}{m+1}+\frac{1}{\frac{(m+1)(m+2)}{2}} = \frac{2m}{m+1}+\frac{2}{(m+1)(m+2)}$$

$$= \frac{2}{m+1}\left\{m+\frac{1}{(m+2)}\right\} = \frac{2}{m+1}\left\{\frac{m^2+2m+1}{(m+2)}\right\}$$

$$= \frac{2}{m+1}\times\frac{(m+1)^2}{m+2} = \frac{2(m+1)}{(m+1)+1} = \text{R.H.S.}$$

So, $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

14. Given, $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$

Now, $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = b^2 + a^2$

$\Rightarrow \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = b^2 + a^2$

$\Rightarrow (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a^2 + b^2$

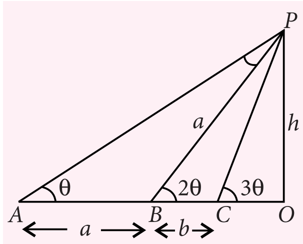
$\Rightarrow 2 + 2\cos(\alpha - \beta) = a^2 + b^2$

$$\Rightarrow \cos(\alpha-\beta) = \frac{a^2+b^2-2}{2}$$

$$\text{Now, } \tan\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}}$$

$$= \pm\sqrt{\frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

15. Let the object be at a height h at P. Let the object when observed from A, B and C, the angles of elevation are θ, 2θ and 3θ respectively.



In ΔPAB, we have

$2\theta = \theta + \angle APB \Rightarrow \angle APB = \theta$

$\therefore \angle PAB = \angle APB = \theta \Rightarrow AB = BP = a$

Similarly, in triangle BPC, $\angle BPC = \theta$

In ΔOPB

$$\sin 2\theta = \frac{h}{a} \Rightarrow h = a\sin 2\theta$$

$\Rightarrow h = 2a\sin\theta\cos\theta \quad ...(i)$

In ΔPBC

$$\frac{PB}{\sin(\pi-3\theta)} = \frac{BC}{\sin\theta} \quad \text{[Using sine rule]}$$

$$\Rightarrow \frac{a}{\sin 3\theta} = \frac{b}{\sin\theta} \Rightarrow \frac{a}{b} = \frac{\sin 3\theta}{\sin\theta}$$

$$\Rightarrow \frac{a}{b} = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} \Rightarrow \frac{a}{b} = 3 - 4\sin^2\theta$$

$$\Rightarrow 4\sin^2\theta = 3 - \frac{a}{b} \Rightarrow \sin^2\theta = \frac{3b-a}{4b}$$

$$\Rightarrow \sin\theta = \sqrt{\frac{3b-a}{4b}} \Rightarrow \cos^2\theta = 1 - \frac{3b-a}{4b} = \frac{a+b}{4b}$$

$$\Rightarrow \cos\theta = \sqrt{\frac{a+b}{4b}}$$

Substituting the values of $\sin\theta$ and $\cos\theta$ in (i), we get

$$h = 2a\sqrt{\frac{3b-a}{4b}}\times\sqrt{\frac{a+b}{4b}} = \frac{a}{2b}\sqrt{(a+b)(3b-a)}$$

MPP-1 MONTHLY Practice Problems

Class XI

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer.

Self check table given at the end will help you to check your readiness.

Sets, Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. The range of the function $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$ is
   (a) $R-\{1\}$ (b) $R-\left\{1,\dfrac{1}{5}\right\}$
   (c) R (d) $R-\left\{\dfrac{1}{5}\right\}$.

2. The domain of definition of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$ is
   (a) $R-\{-1,-2\}$ (b) $(-2,\infty)$
   (c) $R-\{-1,-2,-3\}$ (d) $(-3,\infty)-\{-1,-2\}$.

3. If $A=\left\{(x,y):y=\dfrac{4}{x},0\neq x\in R\right\}$ and $B=\{(x,y):y=x, x>0, x\in R\}$
   (a) $A\cap B=\phi$ (b) $A\cap B$ is singleton set
   (c) $A\cap B$ has two elements
   (d) none of these

4. If $f(x)$ is an even function in R and
   $$f(x)=\begin{cases}-x, & 0\le x\le 1\\ 1, & 1<x<\infty\end{cases}$$
   Then the definition of $f(x)$ in $(-\infty, 0]$ is
   (a) $f(x)=\begin{cases}x, & -1\le x\le 0\\ 1, & -\infty<x<-1\end{cases}$
   (b) $f(x)=\begin{cases}x, & -1\le x\le 0\\ -1, & -\infty<x<-1\end{cases}$
   (c) $f(x)=\begin{cases}-x, & -1\le x\le 0\\ -1, & -\infty<x<-1\end{cases}$
   (d) none of these

5. The range of the function $f(x)=\dfrac{1}{2-\cos 3x}$ is
   (a) $\left[-\dfrac{1}{3},0\right]$ (b) R
   (c) $\left[\dfrac{1}{3},1\right]$ (d) none of these

6. If S in the set of all real x such that $\dfrac{2x-1}{x^3+2x^2+x}>0$, then S contains which of the following open intervals
   (a) $\left(-\dfrac{3}{2},\dfrac{1}{2}\right)$ (b) $\left(-\dfrac{3}{2},-\dfrac{1}{4}\right)$
   (c) $\left(-\dfrac{1}{4},\dfrac{1}{2}\right)$ (d) $\left(\dfrac{1}{2},3\right)$.

One or More Than One Options Correct Type

7. The domain of the function $f(x)=\sqrt{\dfrac{x^2-1}{x-2}}$ is
   (a) $(2,\infty)$ (b) $[1,\infty)$
   (c) $[-1,1]\cup(2,\infty)$ (d) none of these

8. Let $f(x) = 1+\sqrt{x}$ and $g(x)=\dfrac{2x}{x^2+1}$ then
   (a) dom $(f+g)=(-1,\infty)$
   (b) dom $(f+g)=[0,\infty)$
   (c) range of $f\cap$ range of $g=\{1\}$
   (d) range of $f\cup$ range of $g=[-1,\infty)$

9. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $(x, g(x))$ is $\sqrt{3}/4$, then the function $g(x)=$

(a) $\pm\sqrt{1+x^2}$ (b) $\sqrt{1-x^2}$
(c) $-\sqrt{1-x^2}$ (d) $\sqrt{1+x^2}$

10. If $e^x + e^{f(x)} = e$, then for $f(x)$
(a) domain = $(-\infty, 1)$
(b) range = $(-\infty, 1)$
(c) domain = $(-\infty, -1)$
(d) range = $(-\infty, -1)$

11. Let $A = \{1, 2, \{3, 4\}, 5\}$. Which of the following are the subsets of A?
(a) $\{3, 4\}$ (b) $\{2, 5\}$
(c) $\{1, 3, 4\}$ (d) $\{\{3, 4\}\}$

12. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is
(a) 211 (b) 256 (c) 220 (d) 219

13. If $f(x)=\sqrt{x^2-|x|}$, $g(x)=\dfrac{1}{\sqrt{9-x^2}}$, then D_{f+g} contains
(a) $(-3, -1)$ (b) $[1, 3)$
(c) $[-3, 3]$ (d) $\{0\} \cup [1, 3)$.

Comprehension Type

(i) Generally real functions in calculus are described by some formula and their domains are not explicitly stated. In such cases to find the domain of a function. We use the fact that the domain is the set of all real numbers x for which $f(x)$ is a real number.
(ii) The range of a function $f(x)$ is the set of values of $f(x)$ which it attains at points in its domain. For a real function the co-domain is always a subset of R. So, range of a real function f is the set of all points y such that $y = f(x)$, where $x \in \text{dom } f(x)$.

14. Find the domain of definition of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$
(a) $(8, 10)$ (b) $(10, 8)$
(c) $(18, 77)$ (d) $(77, 18)$.

15. Find the range of the function
$$f(x)=\sin\left\{\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$$
(a) $[-1, 0]$ (b) $[1, 2]$
(c) $[-1, 1]$ (d) $[1, 5]$.

Matrix Match Type

16. Consider the set $A = \{1, 2, 3, 4, 5, ..., n\}$. Then

	Column I		Column II
(P)	Number of subsets of A is	(1)	n^n
(Q)	Number of functions that can be defined from A to A is	(2)	2^n
(R)	Number of relations that can be defined on A is	(3)	2^{n^2}
		(4)	$n!$

	P	Q	R
(a)	2	3	1
(b)	2	1	3
(c)	1	2	4
(d)	1	4	2

Integer Answer Type

17. If $f(x)=\cos(\log x)$, then $f(x)f(y)-\dfrac{1}{2}\left[f\left(\dfrac{x}{y}\right)+f(xy)\right]$ has the value

18. If the range of $y=\dfrac{1}{2+\sin 3x+\cos 3x}$ is $\left[\dfrac{1}{n+\sqrt{n}}, \dfrac{1}{n-\sqrt{n}}\right]$, then the value of n is

19. Domain of $f(x) = \sqrt{\log(2x-x^2)}$ is $\{p\}$, then p is equal to

20. Suppose that $A_1, A_2, A_3, ..., A_{30}$ are thirty sets, each containing 6 elements and $B_1, B_2, B_3, ..., B_n$ are n sets, each containing 3 elements. If $\bigcup_{i=1}^{30} A_i = S = \bigcup_{i=1}^{n} B_i$ and each element of S belongs to exactly 10 A_i's and to exactly 9 B_i's then $n = 9p$, where $p =$ ◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......
No. of questions correct ......
Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

CONCEPT BOOSTERS

Class XII

MATRICES AND DETERMINANTS

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION

Rectangular array of mn numbers.

$$A=\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \text{ or } \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{pmatrix}$$

Abbreviated as $A = [a_{ij}]$, $1 \le i \le m$; $1 \le j \le n$, where i denotes the row and j denotes the column, is called a matrix of order $m \times n$.

Special Type of Matrices

- **Row Matrix (or row vectors) :** $A = [a_{11}\, a_{12} \ldots\ldots a_{1n}]$ having one row is called row matrix of order $1\times n$.
- **Column Matrix (or column vectors) :** $A=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ having one column is called column matrix of order $m \times 1$
- **Zero or Null Matrix : ($A = O_{m\times n}$)**

An matrix of order $m \times n$ whose all entries are zero.

$$A=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is a null matrix of order (3×1) and

$$B=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is a null matrix of order (3×3)

- **Horizontal Matrix :** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

  e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$

- **Vertical Matrix :** A matrix of order $m \times n$ is a vertical matrix if $m > n$. *e.g.* $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

- **Square Matrix (Order n) :**

If number of rows = number of columns
Then, the matrix is called square matrix.

Remarks : In a square matrix A = $[a_{ij}]$ the pair of elements a_{ij} and a_{ji} are called **Conjugate Elements.**

- The elements a_{11}, a_{22}, a_{33}, ..., a_{nn} are called **diagonal elements.** The line along which the diagonal elements lie is called **"Principal or Leading"** diagonal.
- The sum of all diagonal entries, Σa_{ii} = trace of the matrix written as, trA.

* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91).
He trains IIT and Olympiad aspirants.

Square Matrix

Triangular Matrix

$$A=\begin{pmatrix}1&3&-2\\0&2&4\\0&0&5\end{pmatrix};$$

Upper Triangular
$a_{ij}=0\,\forall i>j$

$$B=\begin{pmatrix}1&0&0\\2&-3&0\\4&3&3\end{pmatrix}$$

Lower Triangular
$a_{ij}=0\,\forall i<j$

Diagonal Matrix denoted as diag $(d_1, d_2, ..., d_n)$ all elements except the leading diagonal are zero.

Diagonal Matrix

$$\begin{bmatrix}d_1&0&0\\0&d_2&0\\0&0&d_3\end{bmatrix}$$

Unit or Identity Matrix

$$a_{ij}=\begin{bmatrix}1 & if\ \ i=j\\0 & if\ \ i\neq j\end{bmatrix}$$

Note : (1) If $d_1 = d_2 = d_3 = a$ Scalar Matrix
(2) If $d_1 = d_2 = d_3 = 1$ Unit Matrix

Remarks :

- Minimum number of zeroes in a triangular matrix of order $n = n(n - 1)/2$
- Min. number of zeros in a diagonal matrix of order $n = n(n - 1)$

Equality of Matrices

$A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if,

- both have the same order.
- $a_{ij} = b_{ij}$ for each pair of i and j.

ALGEBRA OF MATRICES

Addition : $A + B = [a_{ij} + b_{ij}]$ where A and B are of the same type. (same order)

- **Addition of matrices is commutative.**

i.e. $A + B = B + A$
where A and B are of the same order

- **Matrix addition is associative .**

i.e. $(A + B) + C = A + (B + C)$
where A , B and C are of the same order.

Additive inverse : If $A + B = O = B + A$, where A and B are of same the order. Where A and B are of the same order, then B is called additive inverse of A.

- **Multiplication of A Matrix by A Scalar :**

$$\text{If } A=\begin{bmatrix}a&b&c\\b&c&a\\c&a&b\end{bmatrix} \;;\; kA=\begin{bmatrix}ka&kb&kc\\kb&kc&ka\\kc&ka&kb\end{bmatrix}$$

- **Multiplication of Matrices : (Row by Column)**
AB exists if , $o(A) = m \times n$ and $o(B) = n \times p$
AB exists , but BA does not because $m \neq p$

Note : In the product AB , $\begin{cases}A = \text{pre factor}\\ B = \text{post factor}\end{cases}$

$$A=[a_1, a_2, ...a_n]_{1\times n} \text{ and } B=\begin{bmatrix}b_1\\b_2\\ \vdots\\ b_n\end{bmatrix}_{n\times 1}$$

$AB = [a_1b_1 + a_2b_2 + ... + a_nb_n]_{1\times 1}$
If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ matrix ,

then $(AB)_{ij} = \sum_{r=1}^{n} a_{ir}\cdot b_{rj}$

Properties of Matrix Multiplication

- Matrix multiplication is not commutative .

$$A=\begin{bmatrix}1&1\\0&0\end{bmatrix}; B=\begin{bmatrix}1&0\\0&0\end{bmatrix}; AB=\begin{bmatrix}1&0\\0&0\end{bmatrix}; BA=\begin{bmatrix}1&1\\0&0\end{bmatrix}$$

$\Rightarrow$ $AB \Rightarrow BA$(in general)

- If A and B are two non- zero matrices such that $AB = O$ then A and B are called the divisors of zero. Also $AB = O \Rightarrow |AB| = 0 \Rightarrow |A|\,|B| = 0$

$\Rightarrow$ $|A| = 0$ or $|B| = 0$ but not the converse.

e.g. $AB=\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}-1&1\\1&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

so, $AB = O \not\Rightarrow A = O$ or $B = O$

- If A and B are two matrices such that

(i) $AB = BA \Rightarrow A$ and B commute each other.
(ii) $AB = -BA \Rightarrow A$ and B anti commute each other.

- Matrix Multiplication is Associative

If A , B & C are conformable for the product AB and BC, then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

- Distributivity :

$$\left.\begin{array}{l} A(B+C) = AB + AC \\ (A+B)C = AC + BC \end{array}\right],$$ provided A, B & C are conformable for respective products.

Positive Integral Powers of A Square Matrix

For a square matrix A, $A^2 A = (A A) A = A (A A) = A^3$. Note that for a unit matrix I of any order , $I^m = I$ for all $m \in N$.

Matrix Polynomial

If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n x^0$, then we define a matrix polynomial as

$f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + ... + a_n I^n$

where A is the given square matrix. If $f(A)$ is the null matrix, then A is called the zero or root of the polynomial $f(x)$.

Some Important Definitions

- **Idempotent Matrix :** A square matrix is idempotent provided $A^2 = A$.
  Note that $A^n = A \ \forall \ n \geq 2$, $n \in N$.
- **Periodic Matrix :** A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.
  Note that period of an idempotent matrix is 1.
- **Involutory Matrix :** A square matrix satisfying $A^2 = I$, is said to be an involutory matrix.
  Note : $A = A^{-1}$ for an involutory matrix.

The Transpose of A Matrix :

Let A be any matrix . Then , $A = [a_{ij}]$ of order $m \times n$
$\Rightarrow$ A^T or $A' = [a_{ji}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ of order $n \times m$

Properties of Transpose : If A^T and B^T denote the transpose of A and B ,

- $(A \pm B)^T = A^T \pm B^T$; note that A & B have the same order.
- $(AB)^T = B^T A^T$; A and B are conformable for matrix product AB.
- $(A^T)^T = A$
- $(kA)^T = kA^T$, k is a scalar .

Note : $(A_1 A_2 \ ...... \ A_n)^T = A_n^T ....A_2^T A_1^T$

(reversal law for transpose)

Symmetric & Skew Symmetric Matrix :

A square matrix $A = [a_{ij}]$ is said to be ,

- Symmetric if , $a_{ij} = a_{ji} \ \forall \ i$ & j (conjugate elements are equal)

Remark : Maximum number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

- skew symmetric if ,

$a_{ij} = -a_{ji} \ \forall \ i$ and j (the pair of conjugate elements are additive inverse of each other)

Hence, if A is skew symmetric, then

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall \ i$

Thus, the diagonal elements of a skew symmetric matrix are all zero , but not the converse .

Properties of Symmetric and Skew Symmetric Matrix :

- A is symmetric if $A^T = A$
- A is skew symmetric if $A^T = -A$
- $A + A^T$ is a symmetric matrix.
- $A - A^T$ is a skew symmetric matrix.
- The sum of two symmetric matrices is a symmetric matrix and the sum of two skew symmetric matrices is a skew symmetric matrix .
- If A and B are symmetric matrices, then
  (a) $AB + BA$ is a symmetric matrix.
  (b) $AB - BA$ is a skew symmetric matrix .
- Every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\substack{P \\ \text{Symmetric}}} + \underbrace{\frac{1}{2}(A - A^T)}_{\substack{Q \\ \text{Skew Symmetric}}}$$

Determinants

$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of order two .

Its value is given by : $D = a_1 b_2 - a_2 b_1$

- $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called the determinant of order three .

Its value can be found as :

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \text{ or}$$

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \text{ and so on}$$

In this manner we can expand a determinant in 6 ways using elements of R_1, R_2, R_3 or C_1, C_2 , C_3.

- Condition for the consistency of three simultaneous linear equations in 2 variables.

The lines : $a_1 x + b_1 y + c_1 = 0$ ... (i)
$a_2 x + b_2 y + c_2 = 0$ ... (ii)
$a_3 x + b_3 y + c_3 = 0$ ... (iii)

are concurrent if, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

- Area of a triangle whose vertices are (x_r, y_r) ; $r = 1, 2, 3$ is $D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ If $D = 0$, then the three points are collinear .
- Equation of a straight line passsing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Minors

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element lies. For example,

Let the determinant be $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then minor of a_1 in is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

Hence a determinant of order two will have "4 minors" and a determinant of order three will have "9 minors"

Cofactors

If M_{ij} represents the minor of a_{ij}, then the cofactor is defined as :

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$; where i and j denote the row and column in which the particular element lies. Note that the value of a determinant of order three in terms of 'Minor' and 'Cofactor' can be written as :

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ or

$D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ and so on ...

Adjoint of a Square Matrix

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

Then $(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

Remarks : If A and B are non-singular square matrices of same order, then

- $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj}(AB) = (\text{adj}B)\,(\text{adj}A)$
- $\text{adj}(KA) = K^{n-1}\,(\text{adj}A)$, K is a scalar.

Inverse of a Matrix (Reciprocal Matrix)

A square matrix A is said to be invertible (non-singular) if there exists a matrix B such that, $AB = I = BA$

B is called the inverse (reciprocal) of A and is denoted by A^{-1} .

We have , $A \cdot (\text{adj } A) = |A|\ I_n$

$\Rightarrow A^{-1} A\ (\text{adj } A) = A^{-1} I_n |A|$

$\Rightarrow I_n\ (\text{adj } A) = A^{-1}\ |A|\ I_n$

$\therefore\ A^{-1} = \dfrac{(adj\ A)}{|A|}$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Remarks :

- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$. This is reversal law for inverse.
- If A be an invertible matrix , then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.
- If A is invertible, then
  (i) $(A^{-1})^{-1} = A$;
  (ii) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$, $k \in N$
- If A is an orthogonal matrix, then $AA^T = I = A^TA$ *i.e.*, A square matrix is said to be orthogonal if , $A^{-1} = A^T$.
- $|A^{-1}| = \dfrac{1}{|A|}$

Properties of Determinants

- The value of a determinant remains unaltered, if the rows and columns are interchanged.
  e.g. $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D'$
- If any two rows (or columns) of a determinant be interchanged , the value of determinant is changed in sign only .

e.g. Let $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D'=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $D' = -D$.

- If a determinant has any two rows (or columns) identical, then its value is zero. *e.g.* Let $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then it can be verified that $D = 0$.
- If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

  e.g. If $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D'=\begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

  Then $D' = KD$
- If each element of any row (or column) can be expressed as a sum of two terms then the determinant can be expressed as the sum of two determinants. *e.g.*

$$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column). *e.g.* Let $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

  $D'=\begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3+na_1 & b_3+nb_1 & c_3+nc_1 \end{vmatrix}$. Then $D' = D$

Note :

- While applying this property, atleast one row (or column) must remain unchanged.
- If by putting $x = a$ the value of a determinant vanishes then $(x - a)$ is a factor of the determinant.

Multiplication of two Determinants

- $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix}$

  Similarly two determinants of order three are multiplied.

- If $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then $D^2=\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

  where A_i, B_i, C_i are cofactors

SYSTEM OF EQUATION & CRITERION FOR CONSISTENCY

Gauss – Jordan Method

$$\begin{aligned} a_{11}x+a_{12}y+a_{13}z &= b_1 \\ a_{21}x+a_{22}y+a_{23}z &= b_2 \\ a_{31}x+a_{32}y+a_{33}z &= b_3 \end{aligned} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{31} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\Rightarrow \; AX = B \Rightarrow A^{-1}AX = A^{-1}B$, where $|A| \neq 0$

$\therefore \; X = A^{-1}B = \dfrac{(\text{adj}A)\cdot B}{|A|}$

Remarks :

- If $|A| \neq 0$, system is consistent having unique solution.

  (i) If $|A| \neq 0$ and (adj A)$\cdot B \neq O$ (Null matrix), system is consistent having unique non–trivial solution.

  (ii) If $|A| \neq 0$ and (adj A)$\cdot B = O$ (Null matrix), system is consistent having trivial solution.
- If $|A| = 0$, matrix method fails

If (adj A) $\cdot B = O$ → Consistent (Infinite solutions) or inconsistent

If (adj A) $\cdot B \neq O$ → Inconsistent (no solution)

System of Linear Equation (In Two Variables)

- **Consistent Equations :** Definite & unique solution. [intersecting lines]
- **Inconsistent Equations :** No solution. [Parallel lines]
- **Dependent Equations :** Infinite solutions. [Coincident lines]

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then :

$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2} \Rightarrow$ Given equations are inconsistent.

and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \Rightarrow$ Given equations are coincident.

Cramer's Rule : [Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ...(i) ; $a_2x + b_2y + c_2z = d_2$ ...(ii) ; $a_3x + b_3y + c_3z = d_3$ ...(iii)

Then, $x = \frac{D_1}{D},\ y = \frac{D_2}{D},\ z = \frac{D_3}{D}$

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix};$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}; \text{and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Remarks :

- If $D \neq 0$ and altеast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution .
- If $D \neq 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only .
- If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solutions . In case $\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\}$ represents these parallel planes then also $D = D_1 = D_2 = D_3 = 0$ but the system is inconsistent.
- If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution .

PROBLEMS

Single Correct Answer Type

1. If a, b, c are all different from zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$, then the value of $a^{-1} + b^{-1} + c^{-1}$ is

(a) abc (b) $a^{-1}\,b^{-1}\,c^{-1}$
(c) $-a - b - c$ (d) -1

2. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, A^{-1} is given by

(a) $-A$ (b) A^T
(c) $-A^T$ (d) A

3. If $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, then A^n (where $n \in N$) equals

(a) $\begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & n^2a \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & na \\ 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} n & na \\ 0 & n \end{pmatrix}$

4. If $D = \begin{vmatrix} a^2+1 & ab & ac \\ ba & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$ then $D =$

(a) $1 + a^2 + b^2 + c^2$ (b) $a^2 + b^2 + c^2$
(c) $(a + b + c)^2$ (d) none of these

5. The determinant

$\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$ is

(a) 0 (b) independent of θ
(c) independent of ϕ
(d) independent of θ and ϕ both

6. If $\begin{vmatrix} a+1 & a+2 & a+p \\ a+2 & a+3 & a+q \\ a+3 & a+4 & a+r \end{vmatrix} = 0$, then p, q, r are in :

(a) AP (b) GP
(c) HP (d) none of these

7. Which of the following is an orthogonal matrix

(a) $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$ (b) $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

(c) $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$ (d) $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$

8. The determinant $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} =$

(a) $\frac{1}{3}xyz(x+y)(y+z)(z+x)$

(b) $\frac{1}{4}xyz(x+y-z)(y+z-x)$

(c) $\frac{1}{12}xyz(x-y)(y-z)(z-x)$ (d) none of these

9. If ω is one of the imaginary cube roots of unity, then the value of the determinant $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} =$

(a) 1 (b) 2
(c) 3 (d) none of these

10. Identify the correct statement :

(a) If system of n simultaneous linear equations has a unique solution, then coefficient matrix is singular
(b) If system of n simultaneous linear equations has a unique solution, then coefficient matrix is non singular
(c) If A^{-1} exists, $(\text{adj}A)^{-1}$ may or may not exist
(d) $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $F(x) \cdot F(y) = F(x - y)$

11. A and B are two given matrices such that the order of A is 3×4, if $A'B$ and BA' are both defined then

(a) order of B' is 3×4 (b) order of $B'A$ is 4×4
(c) order of $B'A$ is 3×3 (d) $B'A$ is undefined

12. For a given matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which of the following statements holds good?

(a) $A = A^{-1}\ \forall \theta \in R$
(b) A is symmetric, for $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$
(c) A is an orthogonal matrix, for $\theta \in R$
(d) A is a skew symmetric, for $\theta = n\pi$; $n \in I$

13. Matrix $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, if $xyz = 60$ and $8x + 4y + 3z = 20$, then A (adj A) is equal to

(a) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ (b) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$

(c) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ (d) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

14. For non - zero, real a, b and c $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = \alpha \cdot abc$, then the values of α is

(a) -4 (b) 0 (c) 2 (d) 4

15. Number of triplets of a, b and c for which the system of equations, $ax - by = 2a - b$ and $(c + 1)\,x + cy = 10 - a + 3b$ have infinitely many solutions and $x = 1$, $y = 3$ is one of the solutions, is :

(a) exactly one (b) exactly two
(c) exactly three (d) infinitely many

16. If $A_1, A_3, \ldots, A_{2n-1}$ are n skew symmetric matrices of same order, then $B = \sum_{r=1}^{n} (2r-1)(A_{2r-1})^{2r-1}$ will be

(a) symmetric (b) skew symmetric
(c) neither symmetric nor skew symmetric
(d) data is inadequate

17. If A, B and C are $n \times n$ matrices and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then the value of the $\det(A^2BC^{-1})$ is equal to

(a) $\frac{6}{5}$ (b) $\frac{12}{5}$ (c) $\frac{18}{5}$ (d) $\frac{24}{5}$

Multiple Correct Answer Type

18. Suppose $a_1, a_2, \ldots\ldots$be real numbers, with $a_1 \neq 0$. If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots$are in A.P. then

(a) $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{bmatrix}$ is singular
(b) the system of equations $a_1x + a_2y + a_3z = 0$, $a_4x + a_5y + a_6z = 0$, $a_7x + a_8y + a_9z = 0$ has infinite number of solutions
(c) $B = \begin{bmatrix} a_1 & ia_2 \\ ia_2 & a_1 \end{bmatrix}$ is non-singular ; where $i = \sqrt{-1}$
(d) none of these

19. The determinant $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ is divisible by :

(a) $a + b + c$ (b) $(a + b)\,(b + c)\,(c + a)$
(c) $a^2 + b^2 + c^2$ (d) $(a - b)\,(b - c)\,(c - a)$

20. If A and B are 3×3 matrices and $|A| \neq 0$, then which of the following are true?
(a) $|AB| = 0 \Rightarrow |B| = 0$ (b) $|AB| = 0 \Rightarrow B = 0$
(c) $|A^{-1}| = |A|^{-1}$ (d) $|A + A| = 2\,|A|$

21. The solution(s) of the equation $\begin{vmatrix} x & a & b \\ a & x & a \\ b & b & x \end{vmatrix} = 0$ is/are :
(a) $x = -(a+b)$ (b) $x = a$
(c) $x = b$ (d) $-b$

22. If $\begin{vmatrix} 1 & a & a^2 \\ 1 & x & x^2 \\ b^2 & ab & a^2 \end{vmatrix} = 0$, then
(a) $x = a$ (b) $x = b$ (c) $x = \dfrac{1}{a}$ (d) $x = \dfrac{a}{b}$

23. If p, q, r, s are in A.P. and
$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$
such that $\int_0^2 f(x)dx = -4$ then the common difference of the A.P. can be :
(a) -1 (b) $\dfrac{1}{2}$
(c) 1 (d) none of these

24. If $A(\alpha,\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & e^{\beta} \end{bmatrix}$, then
(a) $A(\alpha, \beta)^T = A(-\alpha, \beta)$
(b) $A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$
(c) $\text{Adj}(A(\alpha, \beta)) = e^{\beta}A(-\alpha, -\beta)$
(d) $A(\alpha, \beta)^T = A(\alpha, -\beta)$

25. The system of equations
$(a\alpha + b)x + ay + bz = 0$, $(b\alpha + c)x + by + cz = 0$
$(a\alpha + b)y + (b\alpha + c)z = 0$
has a non-trival solution , if
(a) a,b,c are in A.P (b) a,b,c are in G.P
(c) a,b,c are in H.P
(d) α is a root of $ax^2 + 2bx + c = 0$

26. Eliminating a,b,c from $x = \dfrac{a}{b-c}, y = \dfrac{b}{c-a}, z = \dfrac{c}{a-b}$ we get
(a) $\begin{vmatrix} 1 & -x & x \\ 1 & -y & y \\ 1 & -z & z \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & -x & x \\ 1 & 1 & -y \\ 1 & z & 1 \end{vmatrix} = 0$
(c) $\begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} = 0$ (d) None of these

Comphension Type

Paragraph for Q. No. 27 to 29

For a given square matrix A, if there exists a matrix B such that $AB = BA = I$, then B is called inverse of A. Every square matrix possesses inverse and it exists if $|A| \neq 0$.
$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} \Rightarrow \text{adj}\,A = |A|(A^{-1})$$

27. Let a matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then it will satisfy the equation
(a) $A^2 - 4A + I = 0$ (b) $A^2 + 4A + I = 0$
(c) $A^2 - 4A - 5I = 0$ (d) $A^2 - 4A + 5I = 0$

28. Let a matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, then A^{-1} will be
(a) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix}$

29. Let matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ satisfies the equation $A^2 + aA + bI = 0$, then the value of $\int_a^{4b} x^3 \cdot \cos x\,dx$ equals
(a) $\dfrac{a+b}{a-b}$ (b) $\dfrac{a-2b}{a-b}$
(c) $\dfrac{a+4b}{4a-b}$ (d) $\dfrac{a-4b}{4a-b}$

Paragraph for Q. No. 30 to 32

Let $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$;
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$;
$a_{n1}x_1 + a_{n2}x_2 + \ldots\, a_{nn}x_n = b_n$ be a system of n linear equations in n unknowns. Then this can be written in the matrix form as
$$AX = B \text{ where } A = [a_{ij}]_{n\times n}; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{pmatrix} B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

Then

(I) If $|A| \neq 0$, the system is consistent, and has a unique solution given by $X = A^{-1}B$

(II) If $|A| = 0$ and $(\text{adj } A)\, B = 0$, then the system is consistent or inconsistent

(III) If $|A| = 0$ and $(\text{adj } A)\, B \neq 0$, then the system is inconsistent.

30. The system of equations $2x - y + 3z = 1, x + y - 2z = 5, x + y + z = -1$ has

(a) a unique solution (b) infinitely many solutions

(c) no solutions (d) none of these

31. Let $2x - y + z = 4, x + 3y + 2z = 12, 3x + 2y + kz = 10$. The value of k in the above system of equations so that system does not have a unique solution is

(a) 2 (b) 3 (c) −1 (d) −2

32. If $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$, the values of λ and μ, for which the system has infinitely many solutions is

(a) $\lambda = 3, \mu = 9$ (b) $\lambda = 3, \mu = 10$

(c) $\lambda = 2, \mu = 10$ (d) $\lambda = 10, \mu = 3$

Matrix-Match Type

33. α, β are the maximum and minimum values of

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

Then match the following :

	Column I		Column II
A.	$\alpha + \beta^{87}$	p.	6
B.	$\alpha^2 - 3\beta^{11}$	q.	2
C.	$f'\left(\dfrac{\pi}{2}\right)$	r.	4
D.	$f\left(\dfrac{\pi}{2}\right)$	s.	−2

34. Let $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

	Column I		Column II
A.	A^{-1}	p.	$\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$
B.	$(\text{adj } A)^{-1}$	q.	$2\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$
C.	$\text{adj}(\text{adj } A)$	r.	$\dfrac{1}{2}\begin{bmatrix} 1+\cos 2x & -\sin 2x \\ \sin 2x & 1+\cos 2x \end{bmatrix}$
D.	$\text{adj}(2A)$	s.	$\dfrac{1}{2}\begin{bmatrix} 1+\cos 2x & \sin 2x \\ -\sin 2x & 1+\cos 2x \end{bmatrix}$

Integer Answer Type

35. $2x + 3y - 3z = 0, 5x - 2y + 2z = 19, x + 7y - 5z = 5$
Find the value of $x + y - z$

36. For what value of $2k/33$ the equations $x + ky + 3z = 0, 3x + ky - 2z = 0, 2x + 3y - 4z = 0$ possess a nontrival solution over the set of rationals ?

37. The value of $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is

38. Let $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ such that $A^TA = I$. Find the value of $x^2 + y^2 + z^2$

39. If $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0, \left(-\pi/4 \leq x \leq \pi/4\right)$ then $\tan x =$

40. Rank of $\begin{bmatrix} 2 & 4 & 7 \\ 17 & 38 & 57 \\ 33 & 70 & 113 \end{bmatrix}$ is

SOLUTIONS

1. (d) : $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ and then open by R_1 to get $ab + abc + ac + bc = 0$; divide it by abc

2. (b) : For Adj A, interchange the diagonal elements and change the sign of counter diagonal elements.

We have $A^{-1} = \dfrac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^T$$

3. (a) : We have $A^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$

$$A^3 = A^2A = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$$

In general by induction, $A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}, \forall n \in N$

4. (a) : Multiply R_1 by a, R_2 by b and R_3 by c and divide the determinant by abc. Now take a, b and c common from C_1, C_2 and C_3. Now use $R_1 \to R_1 + R_2 + R_3$ to get

$$(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}.$$

Now use $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ to get the value of determinant as 1.

5. (b) : Directly open by R_1 to get $\cos^2(\theta + \phi) + \sin^2(\theta + \phi) + \cos 2\phi = 1 + \cos 2\phi$.
Which is independent of θ

6. (a) : Use $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$ and then $C_1 \to C_1 - C_2$ to get

$$\begin{vmatrix} -1 & a+2 & a+p \\ 0 & 1 & q-p \\ 0 & 1 & r-q \end{vmatrix} = 0$$ open by C_1 to get $p + r = 2q$

7. (a) : Matrix $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is orthogonal if

$$\sum a_i^2 = \sum b_i^2 = \sum c_i^2 = 1 \text{ and} \sum a_i\, b_i = \sum b_i\, c_i = \sum c_i\, a_i = 0,$$

$\Rightarrow$ (Option (a) is true.)

8. (c) : $$\begin{vmatrix} x & \dfrac{x(x-1)}{2} & \dfrac{x(x-1)(x-2)}{6} \\ y & \dfrac{y(y-1)}{2} & \dfrac{y(y-1)(y-2)}{6} \\ z & \dfrac{z(z-1)}{2} & \dfrac{z(z-1)(z-2)}{6} \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Use $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ and expand.

9. (c) : Put $\omega^3 = 1$ in $\begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$ and open by R_1 to get $(1 - \omega^2) + (1 - \omega) = 3$

10. (b) : (a) It should be non-singular

(c) Since $A^{-1} = \dfrac{\text{adj}\,A}{|A|}$,

$$\therefore \quad (\text{adj}A)^{-1} = \frac{1}{|\text{adj}A|}\cdot \text{adj}(\text{adj}A) = \frac{A}{|A|}$$

So, its inverse must exist. (d) It should be $F(x + y)$

11. (b) : Order of $A = 3 \times 4$
$\therefore$ Order of $A' = 4 \times 3$
As $A'B$ is defined $\Rightarrow$ let order of $B = 3 \times n$
now $BA' = B_{3 \times n}A'_{4 \times 3} \Rightarrow n = 4$
$\therefore$ order of B is 3×4
$\therefore$ order of $B' = 4 \times 3$
Order of $B'A = 4 \times 4$

12. (c) : A is orthogonal as
$a_{11}^2 + a_{12}^2 = 1 = a_{21}^2 + a_{22}^2$ and $a_{11}a_{21} + a_{22}a_{12} = 0$
For skew symmetric matrix,
$a_{ii} = 0 \;\Rightarrow\; \theta = (2n+1)\dfrac{\pi}{2}, n \in I$
For symmetric matrix, $A = A^T \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi$, $n \in I$

Also $\text{adj}A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $|A| = 1$ hence $A = A^{-1}$ is possible if $\sin\theta = 0$

13. (c) : $A\cdot\text{adj}\ A = |A|\ I$
$|A| = xyz - 8x - 3(z - 8) + 2(2 - 2y)$
$\Rightarrow$ $|A| = xyz - (8x + 3z + 4y) + 28 = 60 - 20 + 28 = 68$

14. (d) $$\frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

Use $R_1 \to R_1 - (R_2 + R_3)$

$$= \frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

Use $R_2 \to R_2 + (1/2)R_1$ and $R_3 \to R_3 + (1/2)R_1$

$$= \frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & c^2 & 0 \\ b^2 & 0 & c^2 \end{vmatrix}$$

$$= \frac{1}{abc}[2b^2(a^2c^2) - 2a^2(-b^2c^2)] = \frac{4a^2b^2c^2}{abc} = 4abc$$

15. (b) : Put $x = 1$ and $y = 3$ in 1st equation
$\Rightarrow$ $a = -2b$ and from 2nd equation
$c = \dfrac{9 + 5b}{4}$; Now use $\dfrac{a}{c+1} = -\dfrac{b}{c} = \dfrac{2a-b}{10-a+3b}$;
From first two equalities $b = 0$ or $c = 1$; if $b = 0$ then $a = 0$ and $c = 9/4$; if $c = 1$ then $b = -1$; $a = 2$

16. (b) : $B = A_1 + 3A_3^3 + ... + (2n-1)(A_{2n-1})^{2n-1}$
$B^T = -[A_1 + 3A_3^3 + ... + (2n-1)(A_{2n-1})^{2n-1}]$
$= -B$. So, skew symmetric

17. (b) : $|A| = 2$, $|B| = 3$, $|C| = 5$

$$\therefore \quad \det(A^2BC^{-1}) = |A^2BC^{-1}| = \frac{|A|^2|B|}{|C|} = \frac{4\cdot 3}{5} = \frac{12}{5}$$

18. (a,b,c) : We have

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 3d & 3d & 3d \\ d & d & d \end{vmatrix} = 0$$

[Using $R_3 \to R_3 - R_2$, and $R_2 \to R_2 - R_1$]

$\Rightarrow \quad A$ is singular

In option (b), the given system of homogeneous equations has infinite number of solutions.

Also $|B| = a_1^2 + a_2^2 \neq 0$. Thus B is non- singular

19. (a, c, d) : Use $C_2 \to C_2 - C_1 - 2C_3$ then $C_1 \to C_1 - C_2$ and then taking $a^2 + b^2 + c^2$ common from first column.

20. (a, c) : For $|AB| = 0 \Rightarrow |A| \cdot |B| = 0 \Rightarrow |A| = 0$ or $|B| = 0$

$AA^{-1} = I \Rightarrow |A|\cdot|A^{-1}| = |I| = 1 \Rightarrow |A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$

21. (a, b, c) : Use $C_1 \to C_1 - C_2$ and then $R_1 \to R_1 + R_2$ to get

$$\begin{vmatrix} 0 & a+x & b+a \\ -(x-a) & x & a \\ 0 & b & x \end{vmatrix} = 0$$. Now expand by C_1 and factorize

22. (a, d) : $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ gives

$$(x-a)(b-1)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & x+a \\ b+1 & a & 0 \end{vmatrix}$$. Expand by C_1 and get the value of $x = a/b$, $x = a$

23. (a, c) : Let $p = a$, $q = a + d$, $r = a + 2d$, $s = a + 3d$

Use $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3 \Rightarrow f(x) = -2d^2$

24. (a, b, c) : We have $A(\alpha, \beta)^T = A(-\alpha, \beta)$

Also, $A(\alpha, \beta)A(-\alpha, -\beta) = I$

$\Rightarrow \quad A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$

Next $\mathrm{Adj}(A(\alpha, \beta)) = |A(\alpha, \beta)|A(\alpha, \beta)^{-1} = e^{\beta}A(-\alpha, -\beta)$

25. (b, d) : The given system of equations will have a non trivial solution if

$$\Delta = \begin{vmatrix} a\alpha+b & a & b \\ b\alpha+c & b & c \\ 0 & a\alpha+b & b\alpha+c \end{vmatrix} = 0$$

$\Rightarrow \quad -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

26. (b, c) : Here $a - xb + xc = 0$, $-ya - b + yc = 0$

$za - zb - c = 0$

Eliminating a,b,c we get $\begin{vmatrix} 1 & -x & x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix} = 0$

Determinants in options (b) and (c) are equal to this determinant

27. (a) : $A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Now, $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

28 (a) : $\because \; A^2 - 4A + I = 0$

$\Rightarrow \quad A^{-1}(A^2) - 4A^{-1}A + A^{-1}I = 0$

$\Rightarrow \quad A = 4I - A^{-1} \Rightarrow A^{-1} = 4I - A$

$\Rightarrow \; A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

29. (c) : Clearly $a = -4$, $b = 1$

$\therefore \int_{-4}^{4} x^3 \cos x dx = 0$

Also $\dfrac{a+4b}{4a-b} = \dfrac{-4+4}{4\times -4-1} = 0$

30. (a) : $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = 2(1 + 2) + 1(1 + 2) + 3(1 - 1) \neq 0$

The solution is unique

31. (b) : If the system does not have a unique solution the value of the determinant of coefficients =0

$$\therefore \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & k \end{vmatrix} = 0 \Rightarrow k = 3$$

32. (b) : The required conditions are $|A| = 0$ and $(\mathrm{Adj}\ A)\ B = 0$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0, \text{ and } \begin{bmatrix} 2\lambda-6 & 2-\lambda & 1 \\ -\lambda+3 & \lambda-1 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{pmatrix} 6 \\ 10 \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e.,$(2\lambda - 6) + (3 - \lambda) + 0 = 0$ and $0\cdot 6 - 10 + \mu = 0$

$\Rightarrow \quad \lambda = 3, \mu = 10$

33. (A) → (r), (B) → (p), (C) → (s), (D) → (q)

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 2+\sin 2x & \cos^2 x & \sin 2x \\ 2+\sin 2x & 1+\cos^2 x & \sin 2x \\ 2+\sin 2x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= (2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2 + \sin 2x)$$

$\therefore$ maximum of $f(x) = \alpha = 3$

minimum of $f(x) = \beta = 1$

$\therefore \alpha + \beta^{87} = 3 + 1 = 4$

$\alpha^2 - 3\beta^{11} = 9 - 3 = 6$

$f'\left(\dfrac{\pi}{2}\right) = 2\cos\pi = -2$

$f\left(\dfrac{\pi}{2}\right) = 2 + 0 = 2$

34. (A) → (r), (B) → (s), (C) → (p), (D) → (q)

$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix} \therefore \text{adj}(A) = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x & -\sin x\cos x \\ \sin x\cos x & \cos^2 x \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos 2x & -\sin 2x \\ \sin 2x & 1+\cos 2x \end{bmatrix}$$

$$\text{adj}(\text{adj}A) = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix} = A$$

$$\therefore (\text{adj}A)^{-1} = \frac{\text{adj}(\text{adj}A)}{|\text{adj}A|} = \frac{1}{2}\begin{bmatrix} 1+\cos 2x & \sin 2x \\ -\sin 2x & 1+\cos 2x \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$\therefore (\text{adj}\,2A) = 2^{2-1}\,\text{adj}(A)$

$$= 2\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

35. (1) : $2x + 3y - 3z = 0$ ...(i)

$5x - 2y + 2z = 19$ ...(ii)

$x + 7y - 5z = 5$ ...(iii)

$5 \times$ (iii) $-$ (ii) $\Rightarrow 37y - 27z = 6$ ...(iv)

$2 \times$ (iii) $-$ (i) $\Rightarrow 11y - 7z = 10$ ...(v)

Solving (iv) & (v) we have, $y = 6$, $z = 8$

$\therefore$ From (i) $x = 3$

So, $x + y - z = 3 + 6 - 8 = 1$

36. (1) : To have non-trivial solution, $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$

$\Rightarrow (-4k + 6) - k(-12 + 4) + 3(9 - 2k) = 0$

$\Rightarrow -4k + 6 + 12k - 4k + 27 - 6k = 0$

$\Rightarrow -2k + 33 = 0 \Rightarrow 2k = 33 \Rightarrow \dfrac{2k}{33} = 1$

37. (0) : $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-b & a(b-c) \end{vmatrix}$

$= 1 \cdot [a(b - c)(b - a) - c(a - b)(c - b)]$

$= (b - c)(b - a)(a - c) = (a - b)(b - c)(c - a)$

Again $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

So the given difference is 0.

38. (1) : $A^T = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

$$\therefore \quad A^T A = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$$

$$= \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix}$$

$\because \quad A^T A = I \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \dfrac{1}{2}$

$3z^2 = 1 \Rightarrow z^2 = \dfrac{1}{3}$ and $6y^2 = 1 \Rightarrow y^2 = \dfrac{1}{6}$

$\therefore \quad x^2 + y^2 + z^2 = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{6} = 1$

39. (1) : On expanding, we get $S^3 + 2C^3 - 3SC^2 = 0$

Where $S = \sin x$, $C = \cos x$

$\Rightarrow (S - C)(S + C)(S - 2C) = 0$

since $-\pi/4 \le x \le \pi/4$, the only solntion is $S - C = 0$ or $\tan x = 1$

40. (2) : $R_3 \to R_3 - R_2 - 8R_1$, R_3 becomes (0, 0, 0)

MPP-1 CLASS XI ANSWER KEY

1.	(b)	2.	(d)	3.	(b)	4.	(a)	5.	(c)
6.	(d)	7.	(c)	8.	(b,c,d)	9.	(b,c)	10.	(a,b)
11.	(b,d)	12.	(d)	13.	(a,b,d)	14.	(a)	15.	(c)
16.	(b)	17.	(0)	18.	(2)	19.	(1)	20.	(6)

CLASS XII **Series 3**

ACE YOUR WAY CBSE

Matrices and Determinants

HIGHLIGHTS

MATRICES

MATRIX

A matrix is a rectangular arrangement of numbers or functions.

- **Order of a matrix :** If a matrix has m rows and n columns, then it is called a matrix of order $m\times n$.
- **Elements :** The numbers or functions occurring in any matrix are called elements of the matrix.

In general, we have, $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = [a_{ij}]_{m\times n}$ is a matrix of order $m\times n$.

Previous Years Analysis						
	2016		2015		2014	
	Delhi	AI	Delhi	AI	Delhi	AI
VSA	3	3	1	1	3	3
SA	1	1	3	3	1	1
LA	1	1	–	–	1	1

Types of Matrices

- **Row Matrix** — A matrix having only one row.
- **Column Matrix** — A matrix having only one column.
- **Square Matrix** — A matrix having equal number of rows and columns *i.e.*, $m = n$
- **Diagonal Matrix** — A square matrix having all its non-diagonal elements zero, *i.e.*, $a_{ij} = 0, \forall\ i \neq j$
- **Scalar Matrix** — A diagonal matrix where all diagonal elements are equal.
- **Identity Matrix** — A diagonal matrix whose all diagonal elements is equal to 1.
- **Zero Matrix** — A matrix whose each and every element is zero.
- **Upper Triangular Matrix** — A square matrix in which $a_{ij} = 0, \forall\ i > j$
- **Lower Triangular Matrix** — A square matrix in which $a_{ij} = 0, \forall\ i < j$

EQUALITY OF MATRICES

Two matrices A and B each of order $m \times n$ are said to be equal if their corresponding elements are equal.

OPERATIONS ON MATRICES

Operations	Definition	Properties
Addition of two matrices	Let A and B be two matrices each of order $m \times n$. Then, $A + B = [a_{ij} + b_{ij}]$ $\forall\ i = 1, 2, ..., m$ and $j = 1, 2, ..., n$	(i) **Commutative law :** For any two matrices A & B, $A + B = B + A$ (ii) **Associative law :** For any three matrices A, B and C, $A + (B + C) = (A + B) + C$ (iii) **Existence of Additive Identity :** Let A be any matrix and O be a zero matrix, then $A + O = A = O + A$ *i.e.* O is the additive identity. (iv) **Existence of Additive Inverse :** For any matrix A, there exists a matrix $(-A)$ such that $A + (-A) = O = (-A) + A$. *i.e.* $(-A)$ is the additive inverse of A.
Multiplication of a matrix by a scalar	Let A be a matrix of order $m \times n$. Then, for any scalar k, $kA = [k \cdot a_{ij}]_{m \times n}$	Let A and B be any two matrices and k and l are scalars, then (i) $k(A + B) = kA + kB$ (ii) $(k + l)\ A = kA + lA$
Multiplication of two matrices	Let A and B be any two matrices of orders $m \times n$ and $n \times p$ respectively. Then $AB = C = [c_{ik}]_{m \times p}$ where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk}$	(i) Multiplication of two matrices is not commutative. (ii) **Associative law :** For any three matrices A, B and C, $(AB)C = A(BC)$ (iii) **Distributive law :** For any three matrices A, B and C, $A(B + C) = AB + AC$, $(A + B)C = AC + BC$ (iv) **Existence of multiplicative identity :** For any square matrix, there exists a matrix I such that $AI = A = IA$ where I is called the identity matrix.

TRANSPOSE OF A MATRIX

For any matrix $A=[a_{ij}]_{m\times n}$, the transpose is obtained by interchanging its rows and columns. It is denoted by A' or A^{T} *i.e.*, $A' = [a_{ji}]_{n\times m}$

PROPERTIES OF TRANSPOSE

- $(A')' = A$
- $(kA)' = kA'$, where k is any constant
- $(A + B)' = A' + B'$
- $(AB)' = B'A'$

SOME SPECIAL MATRICES

- **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if $A = A'$ *i.e.*, $a_{ij} = a_{ji}\ \forall\ i, j$.
- **Skew-symmetric matrix :** A square matrix $A = [a_{ij}]$ is called a skew-symmetric matrix if $A' = -A$ *i.e.*, $a_{ij} = -a_{ji}\ \forall\ i, j$.

Note : (i) All the diagonal elements of a skew-symmetric matrix are zero.

(ii) For any square matrix A, $A + A'$ is symmetric matrix and $A - A'$ is skew- symmetric matrix.

(iii) Any square matrix can be expressed as the sum of symmetric and skew-symmetric matrix, *i.e.*

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

- **Orthogonal matrix :** A square matrix A is called an orthogonal matrix if $AA' = A'A = I$.

ELEMENTARY OPERATION (TRANSFORMATION) OF A MATRIX

(i) Interchange of any two rows (or columns) : If i^{th} row (or column) is interchanged with j^{th} row (or column), we write $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

(ii) Multiplying the elements of a row (or column) by a non-zero scalar : If the elements of i^{th} row (or column) are multiplied by a non-zero scalar k, we write $R_i \to kR_i$ (or $C_i \to kC_i$).

(iii) Adding the elements of a row (or column) and the constant times the corresponding elements of another row (or column) : If k times the elements of j^{th} row (or column) are added to the corresponding elements of the i^{th} row (or column), we write $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$).

INVERTIBLE MATRICES

A square matrix A of order n is said to be invertible if there exists a matrix B of same order such that, $AB = BA = I_n$

Here, B is called the inverse of A and is denoted by A^{-1}.

Note : (i) Inverse of any matrix, if exists, is unique.

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a Matrix by Elementary Operations

Let A be an invertible square matrix of order n.
Then, $A = I_nA = IA$
Now we applying elementary row operation on matrix equation $A = IA$ till L.H.S. becomes I. Then the second factor of R.H.S. *i.e.* A will remain same and first factor will be some matrix B such that $I = BA$
Hence, by definition, $B = A^{-1}$

DETERMINANTS

DETERMINANT

Corresponding to every square matrix A, there exists a number called the determinant of A and denoted by $|A|$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then,

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

PROPERTIES OF DETERMINANTS

- The value of a determinant remains unaltered if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, the value of the determinant is multiplied by -1.
- If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.
- If the elements of a row (or column) of a determinant are multiplied by any scalar, then the value of the new determinant is equal to same scalar times the value of the original determinant.
- If each element of any row (or column) of a determinant is the sum of two numbers, then the determinant is expressible as the sum of two determinants of the same order.

AREA OF A TRIANGLE

Let ABC be a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then area of ΔABC is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

MINORS AND COFACTORS

For any matrix $A = [a_{ij}]_{n\times n}$, if we leave the row and the column of the element a_{ij}, then the determinant thus obtained is called the minor of a_{ij} and it is denoted by M_{ij}.

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij} and denoted by A_{ij}.

$\therefore \quad A_{ij} = (-1)^{i+j} M_{ij}$

ADJOINT OF A MARTRIX

Let $B = [A_{ij}]$ be the matrix of cofactors of matrix $A = [a_{ij}]$. Then the transpose of B is called the adjoint of matrix A.

Note : (i) If $|A| = 0$, then the matrix is singular.

(ii) If $|A| \neq 0$, then the matrix is non-singular.

Properties of adj(A)

- $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I_n$
- $\text{adj } (AB) = (\text{adj } B) \cdot (\text{adj } A)$
- $|\text{adj } A| = |A|^{n-1}$, where n is the order of A.
- $\text{adj } (\text{adj } A) = |A|^{n-2} A \Rightarrow |\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$

INVERSE OF A MATRIX

For any square matrix A, inverse of A is defined as

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

Properties of Inverse

- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Let $AX = B$ be the given system of equations :

(i) If $|A| \neq 0$, the system is consistent and has one unique solution.

(ii) If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system is inconsistent and hence it has no solution.

(iii) If $|A| = 0$ and $(\text{adj } A)B = O$, then the system may be either consistent or inconsistent according as the system has either infinitely many solutions or no solution.

Rvp

PROBLEMS

Very Short Answer Type

1. Simplify :

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

2. Find x, y, z and a for which

$$\begin{bmatrix}x+3 & 2y+x\\ z-1 & 4a-6\end{bmatrix}=\begin{bmatrix}0 & -7\\ 3 & 2a\end{bmatrix}$$

3. If the points $(2, -3)(\lambda, -1)$ and $(0, 4)$ are collinear, find the value of λ.

4. If $A=\begin{bmatrix}3 & -1 & 4\\ 2 & 3 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 3 & 4\\ 2 & 1 & 0\\ -3 & 2 & 3\end{bmatrix}$, find AB.

5. Prove that $\begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix}=0$

Long Answer Type-I

6. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2 - 5A + 7I_2 = O$.

7. If $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\\ 1 & -3 & 1\end{bmatrix}, B=\begin{bmatrix}4 & 5 & 6\\ -1 & 0 & 1\\ 2 & 1 & 2\end{bmatrix}$,

find $4A - 3B$ and $3A - 4B$.

8. Find the inverse of the matrix $A=\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$, using elementary row transformation.

9. Solve $\begin{vmatrix}x-2 & 2x-3 & 3x-4\\ x-4 & 2x-9 & 3x-16\\ x-8 & 2x-27 & 3x-64\end{vmatrix}=0$

10. If $A=\begin{bmatrix}-3\\ 5\\ 2\end{bmatrix}$ and $B = [1 \quad 6 \quad -4]$, then verify that $(AB)' = B'A'$.

Long Answer Type-II

11. Show that :

$$\begin{vmatrix}3a & -a+b & -a+c\\ -b+a & 3b & -b+c\\ -c+a & -c+b & 3c\end{vmatrix}=3(a+b+c)(ab+bc+ca)$$

12. Find the inverse of $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$ and verify that $A^{-1}A = I_3$.

13. A school wants to award its students for the value of Honesty, Regularity and Hardwork with a total cash award of ₹ 6000. Three times the award money for Hardwork added to that given for Honesty amounts to ₹ 11000. The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from the given three values, suggest one more value which the school must include for awards.

14. Show that matrix $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = O$, and hence find A^{-1}.

15. Using elementary row transformation, find the inverse of the matrix $A=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & -1\\ 3 & -5 & 0\end{bmatrix}$.

SOLUTIONS

1. We have,

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ -\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}+\begin{bmatrix}\sin^2\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta & \sin\theta\cos\theta-\sin\theta\cos\theta\\ -\sin\theta\cos\theta+\sin\theta\cos\theta & \cos^2\theta+\sin^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

2. Given $\begin{bmatrix}x+3 & 2y+x\\ z-1 & 4a-6\end{bmatrix}=\begin{bmatrix}0 & -7\\ 3 & 2a\end{bmatrix}$

$\Rightarrow$ $x + 3 = 0, 2y + x = -7$

$z - 1 = 3, 4a - 6 = 2a$

$\therefore$ $x = -3, z = 4, y = -2, a = 3$

3. If the given points are collinear, then

$$\begin{vmatrix} 2 & -3 & 1 \\ \lambda & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $2(-1-4)+3(\lambda-0)+1(4\lambda-0)=0$

$\Rightarrow$ $7\lambda = 10 \Rightarrow \lambda = \frac{10}{7}$

4. $$AB = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 0 \\ -3 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2-12 & 9-1+8 & 12-0+12 \\ 2+6-3 & 6+3+2 & 8+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 16 & 24 \\ 5 & 11 & 11 \end{bmatrix}$$

5. Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Applying $C_3 \to C_3 + C_2$, we get

$$\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

Taking $(a+b+c)$ common from C_3, we get

$$\Delta = (a+b+c)\cdot\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a+b+c) \times 0 = 0$ $\quad [\because C_1$ and C_3 are identical$]$

6. We have, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 7I_2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

7. $$4A - 3B = 4\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ -4 & 0 & 8 \\ 4 & -12 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 15 & 18 \\ -3 & 0 & 3 \\ 6 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -8 & -7 & -6 \\ -1 & 0 & 5 \\ -2 & -15 & -2 \end{bmatrix}$$

Also, $3A - 4B = 3\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - 4\begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 6 & 9 \\ -3 & 0 & 6 \\ 3 & -9 & 3 \end{bmatrix} - \begin{bmatrix} 16 & 20 & 24 \\ -4 & 0 & 4 \\ 8 & 4 & 8 \end{bmatrix} = \begin{bmatrix} -13 & -14 & -15 \\ 1 & 0 & 2 \\ -5 & -13 & -5 \end{bmatrix}$$

8. We know that, $A = IA \Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$

Applying $R_2 \to R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}A$$

Applying $R_1 \to R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

9. We have,

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ we get

$$\begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} = 0$$

$$\Rightarrow \quad (-2)(-6)\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix} = 0$$

Expanding along R_1, we get

$12[(x-2)(7-6)-1(7-3)+2(2-1)]=0$

$\Rightarrow\ 12[(x-2)-4+2]=0 \Rightarrow x=4$

10. We have $A=\begin{bmatrix}-3\\5\\2\end{bmatrix}$ and $B=[1\ \ 6\ \ -4]$

$$\therefore\ AB=\begin{bmatrix}-3\\5\\2\end{bmatrix}[1\ \ 6\ \ -4]=\begin{bmatrix}-3 & -18 & 12\\5 & 30 & -20\\2 & 12 & -8\end{bmatrix}$$

So, $(AB)'=\begin{bmatrix}-3 & 5 & 2\\-18 & 30 & 12\\12 & -20 & -8\end{bmatrix}$

Also, $A'=[-3\ \ 5\ \ 2]$ and $B'=\begin{bmatrix}1\\6\\-4\end{bmatrix}$

$$\therefore\ B'A'=\begin{bmatrix}1\\6\\-4\end{bmatrix}[-3\ \ 5\ \ 2]=\begin{bmatrix}-3 & 5 & 2\\-18 & 30 & 12\\12 & -20 & -8\end{bmatrix}$$

Hence, $(AB)'=B'A'$.

11. Let $\Delta=\begin{vmatrix}3a & -a+b & -a+c\\-b+a & 3b & -b+c\\-c+a & -c+b & 3c\end{vmatrix}$

Applying $C_1\rightarrow C_1+C_2+C_3$, we get

$$\Delta=\begin{vmatrix}a+b+c & -a+b & -a+c\\a+b+c & 3b & -b+c\\a+b+c & -c+b & 3c\end{vmatrix}$$

$$\Rightarrow\ \Delta=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\1 & 3b & -b+c\\1 & -c+b & 3c\end{vmatrix}$$

Applying $R_2\rightarrow R_2-R_1$, $R_3\rightarrow R_3-R_1$, we get

$$\Delta=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\0 & 2b+a & -b+a\\0 & -c+a & 2c+a\end{vmatrix}$$

Expanding along C_1, we get

$\Delta=(a+b+c)\{(2b+a)(2c+a)-(-b+a)(-c+a)\}$

$\Rightarrow\ \Delta=(a+b+c)\{(4bc+2ab+2ca+a^2)$
$-(bc-ab-ac+a^2)\}$

$\Rightarrow\ \Delta=(a+b+c)(3bc+3ab+3ca)$

$\Rightarrow\ \Delta=3(a+b+c)(ab+bc+ca)$

12. We have, $A=\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$

$$\therefore\ |A|=\begin{vmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{vmatrix}=1(16-9)-3(4-3)+3(3-4)$$

$=7-3-3=1\neq 0$

So, A is invertible.

$$\therefore\ \text{Adj } A=\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

Now, $A^{-1}=\dfrac{1}{|A|}\text{adj } A$

$$\Rightarrow\ A^{-1}=\frac{1}{1}\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}=\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

Also, $A^{-1}A=\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$

$$=\begin{bmatrix}7-3-3 & 21-12-9 & 21-9-12\\-1+1+0 & -3+4+0 & -3+3+0\\-1+0+1 & -3+0+3 & -3+0+4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=I_3$$

13. Let the amount of award for Honesty, Regularity and Hardwork be ₹ x, ₹ y and ₹ z respectively. Then,

$x+y+z=6000$ ...(1)

$x+3z=11000$ ... (2)

and $x+z=2y \Rightarrow x-2y+z=0$ ...(3)

Let $A=\begin{bmatrix}1 & 1 & 1\\1 & 0 & 3\\1 & -2 & 1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}6000\\11000\\0\end{bmatrix}$

Then, the matrix equation is $AX=B$.

$\therefore\ X=A^{-1}B$ ...(4)

Now, $|A|=\begin{vmatrix}1 & 1 & 1\\1 & 0 & 3\\1 & -2 & 1\end{vmatrix}=6\neq 0$

$\therefore\ A^{-1}$ exists.

$$\therefore\ (\text{adj } A)=\begin{bmatrix}6 & -3 & 3\\2 & 0 & -2\\-2 & 3 & -1\end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$

$\therefore$ From(4), $X = A^{-1}B = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

$\Rightarrow \quad X = \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 + 0 \\ -12000 + 33000 - 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$

$\Rightarrow \quad x = 500, y = 2000$ and $z = 3500$

Hence, the award money for Honesty, Regularity and Hardwork is ₹ 500, ₹ 2000 and ₹ 3500 respectively. Apart from Honesty, Regularity and Hardwork, the school must include an award for a student to be well-behaved.

14. We have, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Now, $A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad A^2 - 4A - 5I = O$

$\Rightarrow \quad AA - 4A = 5I \Rightarrow (AA) \cdot A^{-1} - 4(A \cdot A^{-1}) = 5I \cdot A^{-1}$

$\Rightarrow \quad A(AA^{-1}) - 4I = 5A^{-1} \Rightarrow AI - 4I = 5A^{-1}$

$\Rightarrow \quad A - 4I = 5A^{-1} \Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

15. We know that, $A = IA$

$$\Rightarrow \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{2}R_2$, we get

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A$$

Applying $R_3 \to (1/4)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

Applying $R_1 \to R_1 + (1/2)R_3$ and $R_2 \to R_2 - (1/2)R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

Hence, $A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$

MPP-1 MONTHLY Practice Problems

Class XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Relations and Functions

Total Marks : 80 **Time Taken : 60 Min.**

Only One Option Correct Type

1. Let $A = \{a, b, c\}$ and $f = \{(a, c), (b, a), (c, b)\}$ be a function from A to A. Then f^{-1} is

(a) $\{(c, a), (a, b), (b, c)\}$
(b) $\{(a, a), (b, b), (c, c)\}$
(c) $\{(a, c), (b, a), (c, b)\}$
(d) none of these

2. Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$, then $fof(x)$ =

(a) $\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$ (b) $\begin{cases}2+x, & 0\le x\le 2\\ 4-x, & 2<x\le 3\end{cases}$

(c) $\begin{cases}2+x, & 0\le x\le 2\\ 2-x, & 2<x\le 3\end{cases}$ (d) none of these

3. Let $f: R \to R$ defined by $f(x)=\dfrac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}$, then

(a) $f(x)$ is one-one but not onto
(b) $f(x)$ is neither one-one nor onto
(c) $f(x)$ is many one but onto
(d) $f(x)$ is one-one and onto

4. The largest interval lying in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ for which the function $f(x)=4^{-x^2}+\cos^{-1}\left(\frac{x}{2}-1\right)+\log(\cos x)$ is defined, is

(a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $\left[-\frac{\pi}{4},\frac{\pi}{2}\right)$ (d) $\left[0,\frac{\pi}{2}\right)$

5. Let f be a real valued function defined by

$f(x)=\dfrac{e^{x}-e^{-|x|}}{e^{x}+e^{|x|}}$, then range of f is

(a) R (b) $[0, 1]$ (c) $[0, 1)$ (d) $[0, 1/2)$

6. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined over the interval $[0, 1]$. The odd extension of $f(x)$ in the interval $[-1, 0]$ is

(a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(c) $-x^2 + x + \sin x - \cos x + \log(1 + |x|)$
(d) none of these

One or More than One Options Correct Type

7. If $f(x)=\dfrac{x}{x^2+1}$ and $f(A)=\left\{y:-\frac{1}{2}\le y<0\right\}$, then set A is

(a) $[-1, 0)$ (b) $(-\infty, -1]$
(c) $(-\infty, 0)$ (d) $(-\infty, \infty)$

8. If the function $f: R \to R$ be such that $f(x) = x - [x]$, where $[\cdot]$ denotes the greatest integer function, then $f^{-1}(x)$ is

Solution Sender of Maths Musing

SET-162

1.	S. Ahamed Thawfeeq	Kerala
2.	N. Jayanthi	Hyderabad
3.	V. Damodhar Reddy	Telangana

SET-161

1.	Khokan Kumar Nandi	WB
2.	Gouri Sankar Adhikary	WB

(a) $\dfrac{1}{x-[x]}$ (b) $[x]-x$
(c) not defined (d) none of these

9. Let $f(x) = [x]^2 + [x+1] - 3$, where $[x] \le x$. Then
(a) $f(x)$ is a many-one and into function
(b) $f(x) = 0$ for infinite number of values of x
(c) $f(x) = 0$ for only two real values
(d) none of these

10. If $f(x) = \cos([\pi^2]x) + \cos([-\pi^2]\, x)$, where $[x]$ stands for the greatest integer function, then
(a) $f\left(\dfrac{\pi}{2}\right) = -1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f\left(\dfrac{\pi}{4}\right) = 1$

11. Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (b, c)\}$ be a relation on A. Here, R is
(a) reflexive (b) symmetric
(c) anti-symmetric (d) transitive

12. The domain of $f(x)$ is $(0, 1)$, therefore domain of $f(e^x) + f(\ln|x|)$ is
(a) $(-1, e)$ (b) $(1, e)$
(c) $(-e, -1)$ (d) $(-e, 1)$

13. The possible values of 'a' for which the function $f(x) = e^{x-[x]} + \cos ax$ (where $[\cdot]$ denotes the greatest integer function) is periodic with finite fundamental period is
(a) π (b) 2π (c) 3π (d) 1

Comprehension Type

Let $f(x) = \dfrac{x^2-x+1}{x^2+x+1}$ for $x \in R$, $g(x) = e^x$ for $x \in R$ and $h(x) = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2.}$

14. Range of the function foh is
(a) $\left(\dfrac{1}{3}, 3\right)$ (b) $\left[\dfrac{1}{3}, 3\right]$
(c) $\left[\dfrac{1}{3}, 1\right]$ (d) $[1, 3]$

15. Range of the function goh is
(a) R (b) $[0, \infty)$ (c) $(-\infty, 0]$ (d) $(0, \infty)$

Matrix Match Type

16. Match the following columns :

	Column I		Column II
(A)	Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, $x \in [0, 2\pi]$ and $g(x) = \max\{1, \lvert x - 1\rvert\}$, $x \in R$, then	(p)	$g(f(1)) = 1$
(B)	Let $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ $\forall\, x \in (-1, 1)$ and $g(x) = \left(\dfrac{3x+x^3}{1+3x^2}\right)$, then	(q)	$f(g(0)) = 0$
(C)	Let $f(x) = 1 + x^2$ and $g(x) = x - x^2$, then	(r)	$f(g(0)) = 1$
		(s)	$g(f(0)) = 1$

Integer Answer Type

17. If $f\left(2x^2 + \dfrac{y^2}{8}, 2x^2 - \dfrac{y^2}{8}\right) = xy$, then $f(60, 48) + f(80, 48) + f(13, 5) = M$. Find the sum of the digits of M.

18. The range of the function $f(x) = \sqrt{(x-6)} + \sqrt{(12-x)}$ is an interval of length $\sqrt{\lambda} - \sqrt{\mu}$, then $\lambda - \mu$ must be

19. Let $n(A) = 4$ and $n(B) = 6$, then the number of one-one functions from A to B is $(6\sqrt{\lambda})^2$, find $\dfrac{\lambda}{2}$.

20. Total number of solutions of the equations $2^x|2 - |x|| = 1$ are

◈◈

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM Set 163

JEE MAIN

1. Let a be a complex number such that $|a| = 1$. If the equation $az^2 + z + 1 = 0$ has a pure imaginary root, then tan (arg a) =

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$

(c) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (d) $\sqrt{\frac{\sqrt{5}+1}{2}}$

2. In triangle ABC, $\tan A$, $\tan B$, $\tan C$ are in A.P. If $\tan A = k$, then $\frac{\sin A \sin C}{\sin B} =$

(a) $\frac{3}{k^2+2}$ (b) $\frac{3}{k^2+3}$

(c) $\frac{3k}{k^2+3}$ (d) $\frac{3k}{k^2+1}$

3. Let $a_1, a_2, a_3, \ldots\ldots$ be a G.P. where $a_1 = a$ and common ratio r are positive integers.

If $\log_8 a_1 + \log_8 a_2 + \ldots.. + \log_8 a_{12} = 2014$, then the number of ordered pairs (a, r) is

(a) 44 (b) 45 (c) 46 (d) 47

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{c} = \hat{i} - \hat{j} + \hat{k}$. If $\vec{r} \times \vec{a} = \vec{b}$ and $\vec{r} \cdot \vec{c} = 3$, then $|\vec{r}| =$

(a) $\sqrt{21}$ (b) $\sqrt{27}$ (c) $\sqrt{29}$ (d) $\sqrt{31}$

5. $\int_0^1 \frac{3dx}{1+x^3} =$

(a) $\ln 2 - \frac{\pi}{\sqrt{3}}$ (b) $\ln 2 + \frac{\pi}{\sqrt{3}}$

(c) $\frac{\pi}{\sqrt{3}}$ (d) $\ln 2$

JEE ADVANCED

6. The slope of tangent to the parabola $(x + 1)^2 = 2y - 1$ drawn from the point $(2, -3)$ is

(a) -2 (b) -1 (c) 6 (d) 7

COMPREHENSION

A straight line through the point (a, b) meets x-axis at A and y-axis at B. O is the origin.

7. If $(a, b) = (4, 1)$, then the minimum value of $OA + OB$ is

(a) 7 (b) 8 (c) 9 (d) 10

8. If $(a, b) = (64, 27)$, then the minimum value of AB is

(a) 120 (b) 125 (c) 130 (d) 132

INTEGER MATCH

9. If $(1 + \sin x)(1 + \cos x) = \frac{5}{4}$, then $(1 - \sin x)(1 - \cos x) = \frac{m}{n} - \sqrt{p}$, where m, n, p are integers with $\frac{m}{n}$, a reduced fraction, then $m + n - p$ is

MATRIX MATCH

10. Match the following.

	List-I		List-II
P.	The distance of the point (3, 0, 5) from the line parallel to the vector $6\hat{i} + \hat{j} - 2\hat{k}$ and passing through the point (8, 3, 1) is	1.	12
Q.	If $z^2 + z + 1 = 0$, then $\sum_{r=1}^{6} (z^r + z^{-r})^2$ is	2.	4
R.	If $f(x)$ is a differentiable function such that $f(0) = 0$, $f(1) = 1$, then the minimum value of $\int_0^1 \left(f'(x)\right)^2 dx$ is	3.	3
S.	The number of solutions of the equation $3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$ in $[0, 2\pi]$ is	4.	1

	P	Q	R	S
(a)	1	4	2	3
(b)	3	1	4	2
(c)	4	2	3	1
(d)	2	3	1	4

See Solution set of Maths Musing 162 on page no. 65

MATH archives

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main and Advanced) and other PETs. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for competitions. In every issue of MT, challenging problems are offered with detailed solutions. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If A is a square matrix such that $A(\text{Adj}\,A)=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$, then $\dfrac{|\text{Adj}(\text{Adj}A)|}{|\text{Adj}A|}$ is equal to

(a) 256 (b) 64 (c) 32 (d) 16

2. If two distinct chords of a parabola $y^2 = 4ax$, passing through $(a, 2a)$ are bisected by the line $x + y = 1$, then length of latus rectum can be

(a) 2 (b) 4 (c) 5 (d) 6

3. The vectors $(al + a'l')\hat{i} + (am + a'm')\hat{j} + (an + a'n')\hat{k}$, $(bl + b'l')\hat{i} + (bm + b'm')\hat{j} + (bn + b'n')\hat{k}$, $(cl + c'l')\hat{i} + (cm + c'm')\hat{j} + (cn + c'n')\hat{k}$

(a) form an equilateral triangle
(b) are coplanar
(c) are collinear (d) are mutually perpendicular

4. If $f(x)=\cos x-\int_0^x (x-t)f(t)\,dt$, then $f''(x) + f(x)$ is equal to

(a) $-\cos x$ (b) $-\sin x$
(c) $\int_0^x (x-t)f(t)\,dt$ (d) zero

5. If $f(x)=\int_1^x \frac{\tan^{-1}t}{t}\,dt,\ x>0$, then the value of $f(e^2)-f\left(\frac{1}{e^2}\right)$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) 2π

6. The number of solutions of the equation $\tan^{-1}\left(\frac{x}{1-x^2}\right)+\tan^{-1}\left(\frac{1}{x^3}\right)=\frac{3\pi}{4}$, belonging to the interval (0, 1) is

(a) 0 (b) 1 (c) 2 (d) infinite

7. With respect to a variable point on the line $x + y = 2a$, chord of contact of the circle $x^2 + y^2 = a^2$ is drawn. If it passes through a fixed point F, the chord of the circle with F as mid point is

(a) parallel to the line $x + y = 2a$
(b) perpendicular to the line $x + y = 2a$
(c) makes angle 45° with the line $x + y = 2a$
(d) none of these

8. The area of the region bounded by the curves $|y + x| \le 1$, $|y - x| \le 1$ and $3x^2 + 3y^2 = 1$ is

(a) $\left(1-\frac{\pi}{3}\right)$ sq. units (b) $\left(2-\frac{\pi}{3}\right)$ sq. units
(c) $\left(3-\frac{\pi}{3}\right)$ sq. units (d) none of these

9. Solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{(1+\ln x+\ln y)^2}$ is

(a) $xy(1+(\ln xy)^2)=\frac{x^2}{2}+C$
(b) $xy(1+\ln xy)=\frac{x^2}{2}+C$
(c) $xy(1+\ln xy)=\frac{x}{2}+C$ (d) none of these

10. P$(t^2, 2t)$, $t \in (0, 1]$ is any arbitrary point on $y^2 = 4x$. Q is the foot of perpendicular drawn from focus S to the tangent drawn at P. Maximum area of triangle PQS is

(a) 1 sq. units (b) 2 sq. units
(c) $\frac{1}{2}$ sq. units (d) 4 sq. units

By : **Prof. Shyam Bhushan,** Director, Narayana IIT Academy, Jamshedpur. Mob. : 09334870021

SOLUTIONS

1. (d) : $A(\text{Adj}\,A) = |A| \cdot I_n$

Clearly $|A| = 4$ and $n = 3$

$$|\text{Adj}(\text{Adj}\,A)| = |A|^{(n-1)^2} = 4^4 = 256$$

$|\text{Adj}A| = |A|^{n-1} = 4^2 = 16$; $\therefore \dfrac{|\text{Adj}\,(\text{Adj}\,A)|}{|\text{Adj}\,A|} = \dfrac{256}{16} = 16$

2. (a) : Any point on the line $x + y = 1$ can be taken as $(t, 1 - t)$

Equation of chord with this point as mid point is

$y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$

It passes through $(a, 2a)$

$\Rightarrow\ t^2 - 2t + 2a^2 - 2a + 1 = 0$

This should have two distinct real roots so $a^2 - a < 0$

$\Rightarrow\ 0 < a < 1$

$\Rightarrow$ length of latus rectum < 4.

3. (b) : Let $\Delta = \begin{vmatrix} al + a'l' & am + a'm' & an + a'n' \\ bl + b'l' & bm + b'm' & bn + b'n' \\ cl + c'l' & cm + c'm' & cn + c'n' \end{vmatrix}$

$$= \begin{vmatrix} a & a' & 0 \\ b & b' & 0 \\ c & c' & 0 \end{vmatrix} \times \begin{vmatrix} l & m & 0 \\ l' & m' & n' \\ 0 & 0 & 0 \end{vmatrix} = 0$$

4. (a) : $f'(x) = -\sin x - \left[xf(x) + \int_0^x f(t)\,dt \right] + xf(x)$

$$= -\sin x - \int_0^x f(t)\,dt$$

$f''(x) = -\cos x - f(x) \Rightarrow f''(x) + f(x) = -\cos x$

5. (c) : $f(x) = \int_1^x \dfrac{\tan^{-1} t}{t}\,dt$

$$\Rightarrow\ f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\tan^{-1} t}{t}\,dt = -\int_1^x \frac{\cot^{-1} t}{t}\,dt$$

$$\Rightarrow\ f(x) - f\left(\frac{1}{x}\right) = \frac{\pi}{2}\log x$$

$$\Rightarrow\ f(e^2) - f\left(\frac{1}{e^2}\right) = \frac{\pi}{2}\log e^2 = \frac{\pi}{2} \times 2 = \pi$$

6. (a) : $\left(\dfrac{x}{1 - x^2}\right) \times \dfrac{1}{x^3} = \left(\dfrac{1}{1 - x^2}\right)\dfrac{1}{x^2} > 1$

So, $\tan^{-1}\left(\dfrac{x}{1 - x^2}\right) + \tan^{-1}\left(\dfrac{1}{x^3}\right)$

$$= \pi + \tan^{-1}\left(\frac{\frac{x}{1 - x^2} + \frac{1}{x^3}}{1 - \frac{1}{x^2(1 - x^2)}}\right)$$

$$= \pi + \tan^{-1}\left(\frac{x^4 + 1 - x^2}{(x^2 - x^4 - 1)x}\right) = \pi + \tan^{-1}\left(-\frac{1}{x}\right) = \frac{3\pi}{4}$$

$\Rightarrow\ x = 1$

7. (a) : Any point on the line $x + y = 2a$ is $(t, 2a - t)$

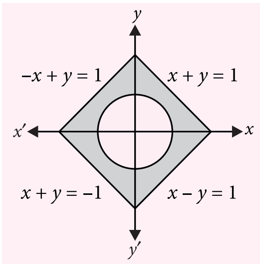
Now equation of chord of contact is $xt + y(2a - t) = a^2$

or $(x - y)t + 2ay - a^2 = 0$

This passes through $F\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$. Now equation of chord with F as mid point is $\dfrac{xa}{2} + \dfrac{ya}{2} = \dfrac{a^2}{2} \Rightarrow x + y = a$

Clearly this is parallel to the line $x + y = 2a$.

8. (b) : Both together form a square of side $\sqrt{2}$ units. $3x^2 + 3y^2 = 1$ is a circle of radius $\dfrac{1}{\sqrt{3}}$ units.



Area of the circle = $\dfrac{\pi}{3}$ sq. units

Area of the square = 2 sq. units.

$\therefore$ Required area $= \left(2 - \dfrac{\pi}{3}\right)$ sq. units

9. (a) : $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{(1 + \ln xy)^2}$

$$\Rightarrow\ xdy + ydx = \frac{xdx}{(1 + \ln xy)^2}$$

$$\Rightarrow\ (1 + \ln xy)^2\, d(xy) = xdx$$

Integrating both sides, we get

$$\int (1 + \ln t)^2 dt = \frac{x^2}{2} + C \quad (\text{Putting } t = xy)$$

$$\Rightarrow\ (1 + \ln t)^2 t - 2\int (1 + \ln t)\, dt = \frac{x^2}{2} + C$$

$$\Rightarrow\ t(1 + \ln t)^2 - 2t - 2(t\ln t - t) = \frac{x^2}{2} + C$$

$$\Rightarrow\ t(1 + \ln t)^2 - 2t\ln t = \frac{x^2}{2} + C$$

$$\Rightarrow\ t(1 + (\ln t)^2) = \frac{x^2}{2} + C$$

$$\Rightarrow\ xy(1 + (\ln xy)^2) = \frac{x^2}{2} + C$$

10. (a) : Equation of PQ is $yt = x + t^2$

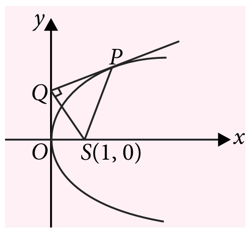
$Q \equiv (0, t)$

$\Rightarrow\ PQ = \sqrt{t^4 + t^2} = t\sqrt{1 + t^2}$

$QS = \sqrt{1 + t^2}$

$\Rightarrow$ Area of $\Delta PQS = \dfrac{1}{2} PQ \times QS$

$$= \frac{1}{2}t(1 + t^2)$$



Which is an increasing function of t

$\therefore$ Max. area of ΔPQS = 1 sq. unit

MATHS MUSING

SOLUTION SET-162

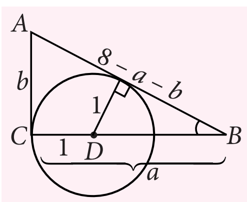
1. **(c) :** $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$
$\Rightarrow\ (2b^2 + a^2 - ca)x^2 + 2b(a + c)x + 2b^2 + c^2 - ca = 0$
Discriminant is $4[ac(a - c)^2 - b^2((a - c)^2 - 4ca) - 4b^4]$
$= 4(ac - b^2)((a - c)^2 + 4b^2) < 0$
since $b^2 - ac > 0$.
The roots are imaginary and distinct.

2. **(b):** $\sum x_i = \sin 2\alpha,\ \sum_{i<j} x_i x_j = \cos 2\alpha$
$\sum_{i<j<k} x_i x_j x_k = \cos\alpha$ and $x_1x_2x_3x_4 = -\sin\alpha$
$\sum_{i=1}^{4} \tan^{-1} x_i = \tan^{-1}\left(\dfrac{\sum x_i - \sum x_i x_j x_k}{1 - \sum x_i x_j + x_1x_2x_3x_4}\right)$
$= \tan^{-1}\left(\dfrac{\sin 2\alpha - \cos\alpha}{1 - \cos 2\alpha - \sin\alpha}\right) = \tan^{-1}(\cot\alpha) = \dfrac{\pi}{2} - \alpha$

3. **(a) :** $|z| = r, |\bar{z}| = |z^4| \Rightarrow r = r^4 \Rightarrow r = 1$
$\therefore\ z = \text{cis}\theta,\ z\bar{z} = z^5 \Rightarrow z^5 = \text{cis}5\theta = 1$
$\therefore\ \cos 5\theta = 1, \sin 5\theta = 0$
$\theta = 0, \pm\dfrac{2\pi}{5}, \pm\dfrac{4\pi}{5}$

4. **(d):** H H H H H H H ......
T H H H H H H H .....
. T H H H H H H H ....
.. T H H H H H H H ...
... T H H H H H H H ..
.... T H H H H H H H.
..... T H H H H H H H
H - head, T-tail, .H or T
The desired probability $= \dfrac{1}{2^7} + \dfrac{6}{2^8} = \dfrac{1}{32}$

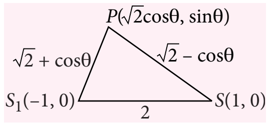
5. **(c) :** $s = 4 \Rightarrow AB = 8 - a - b$



$\sin B = \dfrac{b}{8 - a - b} = \dfrac{1}{a - 1} \Rightarrow a(1 + b) = 8$ ...(i)
$a^2 + b^2 = (8 - a - b)^2 \Rightarrow a(8 - b) = 8(4 - b)$ ...(ii)
(i), (ii) $\Rightarrow b = 2,\ a = \dfrac{8}{3},\ \Delta = \dfrac{8}{3}$

6. **(a, b, c, d) :** $\dfrac{2ab}{a + b} = 2014$
$\Rightarrow\ (a - 1007)(b - 1007) = 1007^2$
$\Rightarrow\ (a - 1007)(b - 1007) = 19^2 \cdot 53^2$
$a - 1007 = 1, b - 1007 = 19^2 \cdot 53^2$
$a - 1007 = 19, b - 1007 = 19 \cdot 53^2$
$a - 1007 = 53, b - 1007 = 19^2 \cdot 53$
$a - 1007 = 19^2, b - 1007 = 53^2$
$\therefore\ a = 1008, 1026, 1060, 1368$

7. **(b):** If (x, y) is the orthocentre, $x = \sqrt{2}\cos\theta$
$\dfrac{y}{\sqrt{2}\cos\theta + 1} = \dfrac{1 - \sqrt{2}\cos\theta}{\sin\theta}$



$\Rightarrow\ y\sin\theta = 1 - 2\cos^2\theta.$
$\therefore\ y^2\left(1 - \dfrac{x^2}{2}\right) = (1 - x^2)^2 \Rightarrow\ y^2 = \dfrac{2(1 - x^2)^2}{2 - x^2}$

8. **(c) :** If $I(x, y)$ is the incentre of triangle PSS_1, then
$x = \cos\theta,\ y = \dfrac{\sin\theta}{\sqrt{2} + 1}$
$\Rightarrow$ Eliminating θ, $\dfrac{x^2}{1} + \dfrac{y^2}{\left(\sqrt{2} - 1\right)^2} = 1$
The length of the latus rectum is
$\dfrac{2b^2}{a} = 2\left(\sqrt{2} - 1\right)^2 = 6 - 4\sqrt{2}$

9. **(6):** $A_1 = 0, A_2 + A_3 = -1 - 3 = -2^2$
$A_4 + A_5 = 6 + 10 = 4^2, \ldots, A_{100} + A_{101} = 100^2$
$\therefore\ S = -2^2 + 4^2 - 6^2 + \ldots - 98^2 + 100^2$
$= -4(1^2 - 2^2 + 3^2 - 4^2 + \ldots + 49^2 - 50^2)$
$= 4(1 + 2 + 3 + \ldots + 50) = 5100$

10. **(a) :** (P) $c^2 - b^2 = a^2 \Rightarrow (c - b)(c + b) = 81$ [$\because a = 9$]
$b = 40, c = 41$ and $b = 12, c = 15$
(Q) $a = 10 \Rightarrow (c - b)(c + b) = 100$
$b = 24, c = 26$
(R) $a = 12 \Rightarrow (c - b)(c + b) = 144$
$(b, c) = (5, 13), (9, 15), (16, 20), (35, 37)$
(S) $a = 20 \Rightarrow (c - b)(c + b) = 400$
$(b, c) = (15, 25), (21, 29), (48, 52), (99, 101)$

MPP-1 CLASS XII ANSWER KEY

1.	(a)	**2.**	(a)	**3.**	(b)	**4.**	(d)	**5.**	(d)
6.	(b)	**7.**	(a,b,c)	**8.**	(c)	**9.**	(a, b)	**10.**	(a, c)
11.	(a, c, d)	**12.**	(c)	**13.**	(a, b, c)	**14.**	(b)	**15.**	(d)
16.	(A)-(p, s); (B)-(q); (C)-(r)					**17.**	(4)	**18.**	(6)
19.	(5)	**20.**	(3)						

WB JEE
SOLVED PAPER 2016

CATEGORY-I (Q. 1 to Q. 50)

Only one answer is correct. Correct answer will fetch full marks 1. Incorrect answer or any combination of more than one answer will fetch -1/4 marks.

1. Let A and B be two events such that $P(A\cap B)=\frac{1}{6}, P(A\cup B)=\frac{31}{45}$ and $P(\bar{B})=\frac{7}{10}$, then
(a) A and B are independent
(b) A and B are mutually exclusive
(c) $P\left(\frac{A}{B}\right)<\frac{1}{6}$ (d) $P\left(\frac{B}{A}\right)<\frac{1}{6}$

2. The value of $\cos 15^\circ \cos\left(7\frac{1}{2}\right)^\circ \sin\left(7\frac{1}{2}\right)^\circ$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

3. The smallest positive root of the equation $\tan x - x = 0$ lies in
(a) $(0, \pi/2)$ (b) $(\pi/2, \pi)$
(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$

4. If in a triangle ABC, AD, BE and CF are the altitudes and R is the circumradius, then the radius of the circumcircle of ΔDEF is
(a) $\frac{R}{2}$ (b) $\frac{2R}{3}$ (c) $\frac{1}{3}R$ (d) none of these

5. The points $(-a, -b)$, (a^2, ab), (a, b), $(0, 0)$ and (a^2, ab), $a \neq 0$, $b \neq 0$ are always
(a) collinear
(b) vertices of a parallelogram
(c) vertices of a rectangle
(d) lie on a circle

6. The line AB cuts off equal intercepts $2a$ from the axes. From any point P on the line AB perpendiculars PR and PS are drawn on the axes. Locus of mid-point of RS is
(a) $x-y=\frac{a}{2}$ (b) $x+y=a$
(c) $x^2+y^2=4a^2$ (d) $x^2-y^2=2a^2$

7. $x + 8y - 22 = 0$, $5x + 2y - 34 = 0$, $2x - 3y + 13 = 0$ are the three sides of a triangle. The area of the triangle is
(a) 36 square unit (b) 19 square unit
(c) 42 square unit (d) 72 square unit

8. The line through the points (a, b) and $(-a, -b)$ passes through the point
(a) $(1, 1)$ (b) $(3a, -2b)$
(c) (a^2, ab) (d) (a, b)

9. The locus of the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=K$ and $\frac{x}{a}-\frac{y}{b}=K$, where K is a non-zero real variable, is given by
(a) a straight line (b) an ellipse
(c) a parabola (d) a hyperbola

10. The equations of a line parallel to the line $3x + 4y = 0$ and touching the circle $x^2 + y^2 = 9$ in the first quadrant is
(a) $3x + 4y = 15$ (b) $3x + 4y = 45$
(c) $3x + 4y = 9$ (d) $3x + 4y = 27$

11. A line passing through the point of intersection of $x + y = 4$ and $x - y = 2$ makes an angle $\tan^{-1}\left(\frac{3}{4}\right)$ with the x-axis. It intersects the parabola $y^2 = 4(x - 3)$ at points (x_1, y_1) and (x_2, y_2) respectively. Then $|x_1 - x_2|$ is equal to
(a) $\frac{16}{9}$ (b) $\frac{32}{9}$ (c) $\frac{40}{9}$ (d) $\frac{80}{9}$

12. The equation of auxiliary circle of the ellipse $16x^2 + 25y^2 + 32x - 100y = 284$ is
(a) $x^2 + y^2 + 2x - 4y - 20 = 0$
(b) $x^2 + y^2 + 2x - 4y = 0$
(c) $(x + 1)^2 + (y - 2)^2 = 400$
(d) $(x + 1)^2 + (y - 2)^2 = 225$

By : Anil Kumar Gupta (akg Classes), Asansol (W.B.) Mob : 09832230099

13. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔOPQ is equilateral, O being the centre. Then the eccentricity e satisfies

(a) $1<e<\frac{2}{\sqrt{3}}$ (b) $e=\frac{2}{\sqrt{2}}$

(c) $e=\frac{\sqrt{3}}{2}$ (d) $e>\frac{2}{\sqrt{3}}$

14. If the vertex of the conic $y^2 - 4y = 4x - 4a$ always lies between the straight lines, $x + y = 3$ and $2x + 2y - 1 = 0$ then

(a) $2 < a < 4$ (b) $-\frac{1}{2}<a<2$

(c) $0 < a < 2$ (d) $-\frac{1}{2}<a<\frac{3}{2}$

15. A straight line joining the points (1, 1, 1) and (0, 0, 0) intersects the plane $2x + 2y + z = 10$ at

(a) (1, 2, 5) (b) (2, 2, 2)
(c) (2, 1, 5) (d) (1, 1, 6)

16. Angle between the planes $x + y + 2z = 6$ and $2x - y + z = 9$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

17. If $y = (1 + x)(1 + x^2)(1+ x^4) ...(1 + x^{2n})$ then the value of $\left(\frac{dy}{dx}\right)$ at $x = 0$ is

(a) 0 (b) −1 (c) 1 (d) 2

18. If $f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = 2$, then $f'(-3)$ equal to

(a) 0 (b) 1 (c) 2 (d) 4

19. $\lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\left(\frac{1-\sqrt{x}}{1-x}\right)}$

(a) is 1 (b) does not exist

(c) is $\sqrt{\frac{2}{3}}$ (d) is ln 2

20. If $f(x)=\tan^{-1}\left[\frac{\log\left(\frac{e}{x^2}\right)}{\log(ex^2)}\right]+\tan^{-1}\left[\frac{3+2\log x}{1-6\log x}\right]$

then the value of $f''(x)$ is

(a) x^2 (b) x (c) 1 (d) 0

21. $\int\frac{\log\sqrt{x}}{3x}dx$ is equal to

(a) $\frac{1}{3}(\log\sqrt{x})^2+c$ (b) $\frac{2}{3}(\log\sqrt{x})^2+c$

(c) $\frac{2}{3}(\log x)^2+c$ (d) $\frac{1}{3}(\log x)^2+c$

22. $\int 2^x(f'(x)+f(x)\log 2)dx$ is equal to

(a) $2^x f'(x) + c$ (b) $2^x \log 2 + c$
(c) $2^x f(x) + c$ (d) $2^x + c$

23. $\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

(a) 1 (b) 0 (c) 2 (d) none of these

24. The value of $\underset{n\to\infty}{\text{Lt}}\left\{\frac{\sqrt{n+1}+\sqrt{n+2}+....+\sqrt{2n-1}}{n^{3/2}}\right\}$ is

(a) $\frac{2}{3}(2\sqrt{2}-1)$ (b) $\frac{2}{3}(\sqrt{2}-1)$

(c) $\frac{2}{3}(\sqrt{2}+1)$ (d) $\frac{2}{3}(2\sqrt{2}+1)$

25. If the solution of the differential equation $x\frac{dy}{dx}+y=xe^x$ be, $xy = e^x\phi(x) + c$, then $\phi(x)$ is equal to

(a) $x + 1$ (b) $x - 1$ (c) $1 - x$ (d) x

26. The order of the differential equation of all parabolas whose axis of symmetry along x-axis is

(a) 2 (b) 3 (c) 1 (d) none of these

27. The line $y = x + \lambda$ is tangent to the ellipse $2x^2 + 3y^2 = 1$ then λ is

(a) −2 (b) 1 (c) $\sqrt{\frac{5}{6}}$ (d) $\sqrt{\frac{2}{3}}$

28. The area enclosed by $y=\sqrt{5-x^2}$ and $y = |x - 1|$ is

(a) $\left(\frac{5\pi}{4}-2\right)$ sq. units (b) $\left(\frac{5\pi-2}{2}\right)$ sq. units

(c) $\left(\frac{5\pi}{4}-\frac{1}{2}\right)$ sq. units (d) $\left(\frac{\pi}{2}-5\right)$ sq. units

29. Let S be the set of points whose abscissas and ordinates are natural numbers. Let $P \in S$ such that the sum of the distance of P from (8, 0) and (0, 12) is minimum among all elements in S. Then the number of such points P in S is

(a) 1 (b) 3 (c) 5 (d) 11

30. Time period T of a simple pendulum of length l is given by $T=2\pi\sqrt{\frac{l}{g}}$. If the length is increased by 2% then an approximate change in the time period is

(a) 2% (b) 1% (c) $\frac{1}{2}\%$ (d) none of these

31. The cosine of the angle between any two diagonals of a cube is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{\sqrt{3}}$

32. If x is a positive real number different from 1 such that $\log_a x$, $\log_b x$, $\log_c x$ are in A.P., then

(a) $b=\frac{a+c}{2}$ (b) $b=\sqrt{ac}$

(c) $c^2=(ac)^{\log_a b}$

(d) none of (a), (b), (c) are correct

33. If a, x are real numbers $|a| < 1$, $|x| < 1$ then $1 + (1 + a)x + (1 + a + a^2)x^2 + ... \infty$ is equal to

(a) $\frac{1}{(1-a)(1-ax)}$ (b) $\frac{1}{(1-a)(1-x)}$

(c) $\frac{1}{(1-x)(1-ax)}$ (d) $\frac{1}{(1-ax)(1-a)}$

34. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(1, 2)$

(c) $(-2, -1)$ (d) none of these

35. The value of $\sum_{n=1}^{13}(i^n + i^{n-1}), i=\sqrt{-1}$, is

(a) i (b) $i - 1$ (c) 1 (d) 0

36. If $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$ and z_1, z_2, z_3 are imaginary numbers, then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1

(c) greater than 1 (d) equal to 3

37. If p, q are the roots of the equation $x^2 + px + q = 0$, then

(a) $p = 1,\ q = -2$ (b) $p = 0, q = 1$

(c) $p = -2, q = 0$ (d) $p = -2, q = 1$

38. The number of values of k for which the equation $x^2 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) are

(a) three

(b) two

(c) infinitely many

(d) no value of k satisfies the requirement

39. The number of ways in which the letters of the word ARRANGE can be permuted such that the R's occur together is

(a) $\frac{\underline{|7}}{\underline{|2}\,\underline{|2}}$ (b) $\frac{\underline{|7}}{\underline{|2}}$ (c) $\frac{\underline{|6}}{\underline{|2}}$ (d) $\underline{|5}\times\underline{|2}$

40. If $\frac{1}{{}^5C_r}+\frac{1}{{}^6C_r}=\frac{1}{{}^4C_r}$, then the value of r equals to

(a) 4 (b) 2 (c) 5 (d) 3

41. For +ve integer n, $n^3 + 2n$ is always divisible by

(a) 3 (b) 7 (c) 5 (d) 6

42. In the expansion of $(x - 1)(x - 2) ....(x - 18)$, the coefficient of x^{17} is

(a) 684 (b) −171 (c) 171 (d) −342

43. $1 + {}^nC_1\cos\theta + {}^nC_1\cos 2\theta + ... + {}^nC_1\cos n\theta$ equals

(a) $\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ (b) $2\cos^2\frac{n\theta}{2}$

(c) $2\cos^{2n}\frac{\theta}{2}$ (d) $\left(2\cos^2\frac{\theta}{2}\right)$

44. If x, y and z be greater than 1, then the value of

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

(a) $\log x\cdot\log y\cdot\log z$ (b) $\log x + \log y + \log z$

(c) 0 (d) $1-\{(\log x)\cdot(\log y)\cdot(\log z)\}$

45. Let A is a 3×3 matrix and B is its adjoint matrix. If $|B| = 64$, then $|A| =$

(a) ± 2 (b) ±4 (c) ±8 (d) ±12

46. Let $Q=\begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$ and $x=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ then Q^3x is equal to

(a) $\begin{pmatrix}0\\1\end{pmatrix}$ (b) $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ (c) $\begin{pmatrix}-1\\0\end{pmatrix}$ (d) $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

47. Let R be a relation defined on the set Z of all integers and xRy when $x + 2y$ is divisible by 3. Then

(a) R is not transitive

(b) R is symmetric only

(c) R is an equivalence relation

(d) R is not an equivalence relation

48. If $A=\{5^n-4n-1 : n\in N\}$ and $B=\{16(n-1) : n\in N\}$, then

(a) $A = B$ (b) $A\cap B=\phi$

(c) $A\subseteq B$ (d) $B\subseteq A$

49. If the function $f : R \to R$ is defined by $f(x) = (x^2+1)^{35}\ \forall\ x \in R$, then f is
(a) one-one but not onto
(b) onto but not one-one
(c) neither one-one nor onto
(d) both one-one and onto

50. Standard deviation of n observations $a_1, a_2, a_3, ..., a_n$ is σ. Then the standard deviation of the observations $\lambda a_1, \lambda a_2, ..., \lambda a_n$ is
(a) $\lambda\sigma$ (b) $-\lambda\sigma$ (c) $|\lambda|\sigma$ (d) $\lambda^n\sigma$

CATEGORY-II (Q. 51 to Q. 65)

Only one answer is correct. Correct answer will fetch full marks 1. Incorrect answer or any combination of more than one answer will fetch –1/2 marks.

51. The locus of the midpoint of chords of the circle $x^2 + y^2 = 1$ which subtends a right angle at the origin is
(a) $x^2+y^2=\frac{1}{4}$ (b) $x^2+y^2=\frac{1}{2}$
(c) $xy = 0$ (d) $x^2 - y^2 = 0$

52. The locus of the midpoints of all chords of the parabola $y^2 = 4ax$ through its vertex is another parabola with directrix is
(a) $x=-a$ (b) $x=a$ (c) $x=0$ (d) $x=-\frac{a}{2}$

53. The $[x]$ denotes the greatest integer, less than or equal to x, then the value of the integral $\int_0^2 x^2[x]dx$ equals
(a) $\frac{5}{3}$ (b) $\frac{7}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

54. The number of points at which the function $f(x) = \max\{a - x, a + x, b\}, -\infty < x < \infty, 0 < a < b$ cannot be differentiable
(a) 0 (b) 1 (c) 2 (d) 3

55. For non-zero vectors $\vec{a}$ and $\vec{b}$ if $|\vec{a}+\vec{b}|<|\vec{a}-\vec{b}|$, then $\vec{a}$ and $\vec{b}$ are
(a) collinear
(b) perpendicular to each other
(c) inclined at an acute angle
(d) inclined at an obtuse angle

56. General solution of $y\frac{dy}{dx}+by^2 = a\cos x, 0 < x < 1$ is
(a) $y^2 = 2a(2b\sin x + \cos x) + ce^{-2bx}$
(b) $(4b^2 + 1)y^2 = 2a(\sin x + 2b\cos x) + ce^{-2bx}$
(c) $(4b^2 + 1)y^2 = 2a(\sin x + 2b\cos x) + ce^{2bx}$
(d) $y^2 = 2a(2b\sin x + \cos x) + ce^{-2bx}$

57. The points of the ellipse $16x^2 + 9y^2 = 400$ at which the ordinate decreases at the same rate at which the abscissa increases is/are given by
(a) $\left(3,\frac{16}{3}\right)$ and $\left(-3,\frac{-16}{3}\right)$
(b) $\left(3,\frac{-16}{3}\right)$ and $\left(-3,\frac{16}{3}\right)$
(c) $\left(\frac{1}{16},\frac{1}{9}\right)$ and $\left(-\frac{1}{16},-\frac{1}{9}\right)$
(d) $\left(\frac{1}{16},-\frac{1}{9}\right)$ and $\left(-\frac{1}{16},\frac{1}{9}\right)$

58. The letters of the word COCHIN are permuted and all permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word COCHIN is
(a) 96 (b) 48 (c) 183 (d) 267

59. If the matrix $A=\begin{pmatrix}2&0&0\\0&2&0\\2&0&2\end{pmatrix}$, then $A^n=\begin{pmatrix}a&0&0\\0&a&0\\b&0&a\end{pmatrix}, n\in N$ where
(a) $a = 2n, b = 2^n$ (b) $a = 2^n, b = 2n$
(c) $a = 2^n, b = n2^{n-1}$ (d) $a = 2^n, b = n2^n$

60. The sum of n terms of the following series, $1^3 + 3^3 + 5^3 + 7^3 + ...$ is
(a) $n^2(2n^2 - 1)$ (b) $n^3(n - 1)$
(c) $n^3 + 8n + 4$ (d) $2n^4 + 3n^2$

61. If α and β are roots of $ax^2 + bx + c = 0$ then the equation whose roots are α^2 and β^2 is
(a) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$
(b) $a^2x^2 + (b^2 - 2ac)x + c^2 = 0$
(c) $a^2x^2 + (b^2 + ac)x + c^2 = 0$
(d) $a^2x^2 + (b^2 + 2ac)x + c^2 = 0$

62. If ω is an imaginary cube root of unity, then the value of $(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + ...$
$.... + (n - 1)(n - \omega)(n - \omega^2)$ is
(a) $\frac{n^2}{4}(n+1)^2-n$ (b) $\frac{n^2}{4}(n+1)^2+n$
(c) $\frac{n^2}{4}(n+1)^2$ (d) $\frac{n^2}{4}(n+1)^2-n$

63. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then the value of ${}^nC_8$ is
(a) 10 (b) 7 (c) 9 (d) 8

64. In a group of 14 males and 6 females, 8 and 3 of the males and the females respectively are aged above 40 years. The probability that a person selected at random from the group is aged above 40 years, given that the selected person is a female, is

(a) $\frac{2}{7}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{5}{6}$

65. The equation $x^3 - yx^2 + x - y = 0$ represents
(a) a hyperbola and two straight lines
(b) a straight line
(c) a parabola and two straight lines
(d) a straight line and a circle

CATEGORY-III (Q. 66 to Q. 75)

One ore more answer(s) is (are) correct. Correct answer(s) will fetch full marks 2. Any combination containing one or more incorrect answer will fetch 0 marks. If all correct answers are not marked and also no incorrect answer is marked then score = 2 × number of correct answers marked/ actual number of correct answers.

66. If the first and the $(2n + 1)^{th}$ terms of an AP, GP and HP are equal and their n^{th} terms are respectively a, b, c then always
(a) $a = b = c$ (b) $a \geq b \geq c$
(c) $a + c = b$ (d) $ac - b^2 = 0$

67. The coordinates of a point on the line $x + y + 1 = 0$ which is at a distance $\frac{1}{5}$ unit from the line $3x + 4y + 2 = 0$ are
(a) $(2, -3)$ (b) $(-3, 2)$ (c) $(0, -1)$ (d) $(-1, 0)$

68. If the parabola $x^2 = ay$ makes an intercept of length $\sqrt{40}$ unit on the line $y - 2x = 1$ then a is equal to
(a) 1 (b) –2 (c) –1 (d) 2

69. If $f(x)$ is a function such that $f'(x) = (x - 1)^2 (4 - x)$, then
(a) $f(0) = 0$
(b) $f(x)$ is increasing in $(0, 3)$
(c) $x = 4$ is a critical point of $f(x)$
(d) $f(x)$ is decreasing in $(3, 5)$

70. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

71. If $\varphi(t) = \begin{cases} 1, & \text{for } 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$ then

$$\int_{-3000}^{3000} \left(\sum_{r'=2014}^{2016} \varphi(t - r')\varphi(t - 2016) \right) dt =$$

(a) a real number (b) 1
(c) 0 (d) does not exist

72. If the equation $x^2 + y^2 - 10x + 21 = 0$ has real roots $x = \alpha$ and $y = \beta$ then
(a) $3 \leq x \leq 7$ (b) $3 \leq y \leq 7$
(c) $-2 \leq y \leq 2$ (d) $-2 \leq x \leq 2$

73. If $z = \sin\theta - i\cos\theta$ then for any integer n

(a) $z^n + \frac{1}{z^n} = 2\cos\left(\frac{n\pi}{2} - n\theta\right)$

(b) $z^n + \frac{1}{z^n} = 2\sin\left(\frac{n\pi}{2} - n\theta\right)$

(c) $z^n - \frac{1}{z^n} = 2i\sin\left(n\theta - \frac{n\pi}{2}\right)$

(d) $z^n - \frac{1}{z^n} = 2i\cos\left(\frac{n\pi}{2} - n\theta\right)$

74. Let $f: X \to X$ be such that $f(f(x)) = x$ for all $x \in X$ and $X \subseteq R$, then
(a) f is one-to-one
(b) f is onto
(c) f is one-to-one but not onto
(d) f is onto but not one-to-one

75. If A, B are two events such that $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ then

(a) $P(A) + P(B) \leq \frac{11}{8}$ (b) $P(A) \cdot P(B) \leq \frac{3}{8}$

(c) $P(A) + P(B) \geq \frac{7}{8}$ (d) none of these

SOLUTIONS

1. **(a) :** $\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \frac{31}{45} = P(A) + 1 - P(\bar{B}) - \frac{1}{6}$

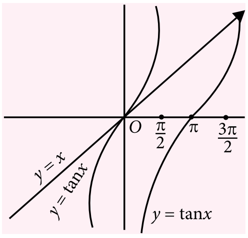
$\Rightarrow P(A) = \frac{31}{45} - \frac{5}{6} + \frac{7}{10} = \frac{5}{9}$

$\because P(A \cap B) = \frac{1}{6} = \frac{5}{9} \times \frac{3}{10} = P(A)P(B)$

$\therefore$ A and B are independent

2. **(b) :** We have, $\frac{1}{2}\cos 15° \sin 15° = \frac{1}{4} \cdot \sin 30° = \frac{1}{8}$

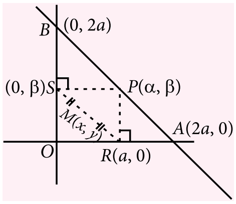
3. **(c) :** We take, $y = \tan x$ and $y = x$.
From their graph, it is clear that smallest positive root lies in $\left(\pi, \frac{3\pi}{2}\right)$



4. **(a) :** Here, ΔDEF is a pedal triangle of ΔABC
$\therefore$ Radius of the circumcircle of ΔDEF = Half of the radius of the circumcircle of $\Delta ABC = \frac{R}{2}$

5. **(a) :** $\because$ Slopes of join of any 2 given points $= \frac{b}{a}$
$\therefore$ points are collinear

6. **(b) :** Let $P(\alpha, \beta)$ be any point on



$$AB \equiv \frac{x}{2a} + \frac{y}{2a} = 1$$

$\Rightarrow \quad x + y = 2a$

$\therefore \quad \alpha + \beta = 2a \qquad$ ...(i)

Let M be the mid point of $R(\alpha, 0)$ and $S(0, \beta)$

$\therefore \quad x = \frac{\alpha + 0}{2}, \quad y = \frac{0 + \beta}{2} \Rightarrow \alpha = 2x, \quad \beta = 2y$

From (i), $2x + 2y = 2a \Rightarrow x + y = a$

7. **(b) :** On solving, vertices are (6, 2), (–2, 3) and (4, 7)

$$\therefore \quad \text{Area} = \frac{1}{2}\begin{vmatrix} 6-(-2) & 2-3 \\ 6-4 & 2-7 \end{vmatrix} = 19 \text{ sq. units}$$

8. **(c, d)**

9. **(d) :** We have, $\frac{x}{a} + \frac{y}{b} = K \qquad$ ...(i)

$\frac{x}{a} - \frac{y}{b} = K \qquad$ ...(ii)

On multiplying (i) and (ii), we get

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = K^2 \Rightarrow \frac{x^2}{(Ka)^2} - \frac{y^2}{(Kb)^2} = 1$ which is a hyperbola

10. **(a) :** Line parallel to $3x + 4y = 0$ is $3x + 4y + k = 0 \qquad$ ...(i)
Centre and radius of $x^2 + y^2 = 9 \qquad$ ...(ii)
are (0, 0) and 3 units respectively.

If (i) be a tangent to (ii) then $\frac{0 + 0 + k}{\sqrt{3^2 + 4^2}} = \pm 3 \Rightarrow k = \pm 15$

For tangent in 1st quadrant, $k = -15$
$\therefore$ Tangent is $3x + 4y = 15$

11. **(b) :** Point of intersection of lines $x + y = 4$ and $x - y = 2$ is (3, 1)

Line through (3, 1) and making angle $\tan^{-1}\left(\frac{3}{4}\right)$ with x-axis is $y - 1 = \frac{3}{4}(x - 3) \Rightarrow y = \frac{3x - 5}{4} \qquad$ ...(i)

On solving (i) with $y^2 = 4(x - 3)$, we get

$9x^2 - 30x + 25 = 64(x - 3) \Rightarrow 9x^2 - 94x + 217 = 0$
According to question, its roots are x_1, x_2

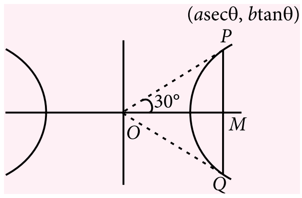
$\therefore \quad x_1 + x_2 = \frac{94}{9}; \; x_1 x_2 = \frac{217}{9}$

$\therefore \quad |x_1 - x_2| = \sqrt{(x_1 + x_2)^2 - 4x_1 x_2} = \frac{32}{9}$

12. **(a) :** Ellipse is $\frac{(x+1)^2}{5^2} + \frac{(y-2)^2}{4^2} = 1$
Centre is (–1, 2) and $a = 5$, $b = 4$ $(a > b)$
$\therefore$ Auxiliary circle is $(x + 1)^2 + (y - 2)^2 = 5^2$
$\Rightarrow \quad x^2 + y^2 + 2x - 4y - 20 = 0$

13. **(d) :** $OM = a\sec\theta$, $PM = b\tan\theta$

$\tan 30° = \frac{PM}{OM} = \frac{b \sin\theta}{a}$



$\sin\theta = \frac{a}{b\sqrt{3}}$

$\Rightarrow \quad \sin^2\theta = \frac{a^2}{3b^2}$

$\because \quad 0 < \sin^2\theta < 1$

$\therefore \quad 0 < \frac{a^2}{3b^2} < 1 \quad \left[\because \frac{b^2}{a^2} = e^2 - 1\right]$

$\Rightarrow \quad \frac{3b^2}{a^2} > 1 \Rightarrow 3(e^2 - 1) > 1 \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$

14. **(b) :** We have, $(y - 2)^2 = 4(x - a + 1)$, Vertex is $(a - 1, 2)$
$\because \quad (a - 1, 2)$ lies between parallel lines $x + y = 3$ and $2x + 2y - 1 = 0$
$\therefore \quad (a - 1) + 2 - 3$ and $2(a - 1) + 2(2) - 1$ must be of opposite signs.

$\Rightarrow \quad (a - 2)(2a + 1) < 0 \Rightarrow -\frac{1}{2} < a < 2$

15. **(b) :** Line joining (1, 1, 1) and (0, 0, 0) is

$\frac{x - 0}{1 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{1 - 0}$ *i.e.*, $\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = k(\text{say}) \qquad$ ...(i)

$\therefore$ Any point on (i) may be $P(k, k, k)$
If $P(k, k, k)$ lies on $2x + 2y + z = 10$ then $5k = 10$
$\Rightarrow \quad k = 2 \quad \therefore \; P$ is (2, 2, 2).

16. **(c) :** $\theta = \cos^{-1}\left|\frac{1(2) + 1(-1) + 2(1)}{\sqrt{1+1+4}\sqrt{4+1+1}}\right| = \cos^{-1}\left|\frac{3}{6}\right| = \frac{\pi}{3}$

17. **(c) :** We have, $y = \frac{(1-x)(1+x)(1+x^2)...(1+x^{2n})}{(1-x)}$

$$= \frac{1 - x^{4n}}{1 - x}$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{(1-x)(-4nx^{4n-1}) - (1 - x^{4n})(-1)}{(1-x)^2}$

$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{x=0} = 1$

18. (c) : $\because$ $f(x)$ is an odd differentiable function .

$\therefore$ $f'(x)$ will be an even function.

$\Rightarrow$ $f'(-3) = f'(3) = 2$

19. (c) : $\lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{\frac{(1-\sqrt{x})}{(1+\sqrt{x})(1-\sqrt{x})}} = \lim\limits_{x\to 1}\left(\dfrac{1+x}{2+x}\right)^{\frac{1}{1+\sqrt{x}}} = \sqrt{\dfrac{2}{3}}$

20. (d) : $f(x) = \tan^{-1}\left(\dfrac{1-\log x^2}{1+\log x^2}\right) + \tan^{-1}\left(\dfrac{3+2\log x}{1-3(2\log x)}\right)$

$= \tan^{-1}1 - \tan^{-1}(\log x^2) + \tan^{-1}3 + \tan^{-1}(2\log x)$

$= \tan^{-1}1 + \tan^{-1}3$

$f'(x) = 0 \Rightarrow f''(x) = 0$

21. (a) : $I = \dfrac{1}{6}\displaystyle\int \dfrac{\log x}{x}dx = \dfrac{1}{6}\cdot\dfrac{(\log x)^2}{2} + c = \dfrac{1}{3}(\log\sqrt{x})^2 + c$

22. (c): $I = 2^x\cdot f(x) - \displaystyle\int (2^x\log 2) f(x)dx + \int 2^x\cdot f(x)\cdot\log 2dx$

$= 2^x f(x) + c$

23. (b) : Let $I = \displaystyle\int_0^1 \log\left(\frac{1-x}{x}\right)dx = \int_0^1 \log\left\{\frac{1-(1-x)}{1-x}\right\}dx$

$= \displaystyle\int_0^1 \log\left(\frac{x}{1-x}\right)dx = -\int_0^1 \log\left(\frac{1-x}{x}\right)dx = -I$

$\therefore$ $2I = 0 \Rightarrow I = 0$

24. (a) : $\underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{n}\left\{\sqrt{1+\dfrac{1}{n}} + \sqrt{1+\dfrac{2}{n}} + \ldots. + \sqrt{1+\dfrac{n-1}{n}}\right\}$

$= \underset{n\to\infty}{\text{Lt}}\ \dfrac{1}{n}\displaystyle\sum_{r=1}^{n-1}\sqrt{1+\frac{r}{n}} = \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}\Big[(1+x)^{3/2}\Big]_0^1$

$= \dfrac{2}{3}(2^{\frac{3}{2}} - 1) = \dfrac{2}{3}(2\sqrt{2} - 1)$

25. (b) : $x\dfrac{dy}{dx} + y = xe^x \Rightarrow \dfrac{d}{dx}(xy) = xe^x$

On integrating, $xy = x(e^x) - \displaystyle\int 1\cdot e^x dx = e^x(x-1) + c$

On comparing, we get $\phi(x) = x - 1$

26. (a) : Such parabola will be of the form

$(y - 0)^2 = 4a(x - \alpha)$

This involves 2 constants (a and α)

$\therefore$ Order of the diff. eqn. formed will be 2

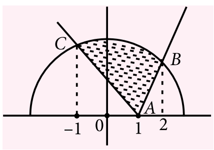
27. (c) : $y = x + \lambda \Rightarrow m = 1, c = \lambda$

Also, $\dfrac{x^2}{1/2} + \dfrac{y^2}{1/3} = 1 \Rightarrow a^2 = \dfrac{1}{2}, b^2 = \dfrac{1}{3}$

From condition, $c^2 = a^2m^2 + b^2$.

We get $\lambda^2 = \dfrac{1}{2} + \dfrac{1}{3} = \dfrac{5}{6}$

28. (c) : Reqd. area $ABCA$



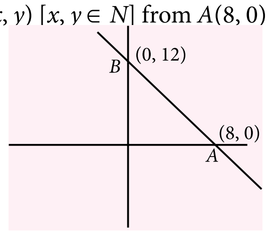
$= \displaystyle\int_{-1}^{2}\sqrt{5-x^2}\,dx - \int_{-1}^{2}|x-1|\,dx$

$= \displaystyle\int_{-1}^{2}\sqrt{(\sqrt{5})^2 - x^2}\,dx - \int_{-1}^{1}(1-x)dx - \int_{1}^{2}(x-1)dx$

$= \left(\dfrac{5\pi}{4} - \dfrac{1}{2}\right)$ sq. units

29. (b) : $AB \equiv \dfrac{x}{8} + \dfrac{y}{12} = 1$

As sum of the distances of $P(x, y)$ $[x, y \in N]$ from $A(8, 0)$ to $B(0, 12)$ should be minimum, P must lie on AB



Possible points $x = 2, y = 9$

$x = 4, y = 6$

$x = 6, y = 3$

30. (b) : $\log T = \log 2\pi + \dfrac{1}{2}(\log l - \log g)$

Differentiating both sides, we get $\dfrac{1}{T}\cdot dT = \dfrac{1}{2}\cdot\dfrac{1}{l}\cdot dl$

$\Rightarrow$ $\dfrac{dT}{T}\times 100 = \dfrac{1}{2}\cdot\left(\dfrac{dl}{l}\times 100\right) = \dfrac{1}{2}\times 2 = 1\%$

31. (a) : We consider a cube with one vertex origin length of each edge = 1 unit. A diagonal joining (0, 0, 0) and (1, 1, 1) has d.c's $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$

Another diagonal joining (0, 0, 1) and (1, 1, 0) has d.c's $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{-1}{\sqrt{3}}\right)$

$\therefore$ $\cos\theta = \dfrac{1}{\sqrt{3}}\cdot\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{3}}\cdot\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{3}}\left(-\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{3}$

32. (c) : $2\log_b x = \log_a x + \log_c x = \dfrac{1}{\log_x a} + \dfrac{1}{\log_x c}$

$= \dfrac{\log_x a + \log_x c}{\log_x a\cdot\log_x c}$

$\Rightarrow$ $\dfrac{2}{\log_x b} = \log_x ac\cdot\log_a x\cdot\log_c x = \log_a ac\cdot\log_c x$

$\Rightarrow$ $2 = \log_a ca\cdot\log_c x\cdot\log_x b = \log_a ca\cdot\log_c b$

$\Rightarrow$ $2 = \log_c(b^{\log_a ca})$

$\Rightarrow$ $c^2 = b^{\log_a ac} = (ac)^{\log_a b}$ $\quad [\because b^{\log_x a} = a^{\log_x b}]$

33. (c) : $\dfrac{1}{1-a}[(1-a) + (1-a^2)x + (1-a^3)x^2 + \ldots. \text{ to } \infty]$

$= \dfrac{1}{1-a}[(1 + x + x^2 + \ldots \text{ to } \infty) - (a + a^2x + a^3x^2 + \ldots \text{ to } \infty]$

$= \dfrac{1}{1-a}\left[\dfrac{1}{1-x} - \dfrac{a}{1-ax}\right] = \dfrac{1}{1-a}\left[\dfrac{1-ax-a+ax}{(1-x)(1-ax)}\right]$

$= \dfrac{1}{(1-x)(1-ax)}$

34. (a) : $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \quad \log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow \quad \frac{1}{2}\log_{0.3}(x-1) < 0 \Rightarrow (x-1) > (0.3)^0 \Rightarrow x > 2$

35. (*) : $\sum_{n=1}^{13}(i^n + i^{n-1}) = \sum_{n=1}^{13} i^n + \sum_{n=1}^{13} i^{n-1}$

$= (i + i^2 + i^3 + ... + i^{13}) + (i^0 + i + i^2 + i^3 + ... + i^{12})$

$= i + 1$

[$\because$ Sum of any 4 consecutive powers of i is zero]

$\therefore$ None of the options is correct.

36. (a) : $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ ...(i)

$\because \quad |z|^2 = z \cdot \bar{z} = 1 \Rightarrow z = \frac{1}{\bar{z}}$

$\therefore \quad |z_1 + z_2 + z_3| = \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right| = \left|\overline{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}}\right|$

$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ [from (i)]

37. (a) : $p + q = -p$ and $p \cdot q = q$

$\Rightarrow \quad 2p + q = 0$ and $q = 0$ or $p = 1$

When $q = 0$, $p = 0$ (not possible)

When $p = 1$, $q = -2$

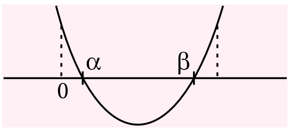
38. (c) : $f(0) \cdot f(1) > 0$

$\Rightarrow \quad k \cdot (1 - 3 + k) > 0$

$\Rightarrow \quad k(k - 2) > 0$

$\Rightarrow \quad k < 0$ or $k > 2$

$\therefore$ Number of values of k is infinite.



39. (c) : Required number of ways $= \frac{6!}{2!}$

40. (b) : $\frac{r!\,(5-r)!}{5!} + \frac{r!(6-r)!}{6!} = \frac{r!\,(4-r)!}{4!}$

$\Rightarrow \quad 6(5 - r) + (6 - r)(5 - r) = 6 \times 5$

$\Rightarrow \quad r^2 - 17r + 30 = 0 \Rightarrow r = 2, 15$

$\because \quad r \neq 15, \therefore r = 2$

41. (a) : $n^3 + 2n = 3, 12, 33, ...$ etc. for $n = 1, 2, 3, ...$

$\therefore \quad n^3 + 2n$ is always divisible by 3.

42. (b) : Required coefficient of x^{17}

$= -(1 + 2 + 3 + ... + 18) = \frac{-18 \times 19}{2} = -171$

43. (a) : $\because e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{in}\theta = \cos n\theta + i\sin n\theta$

$(1 + e^i\theta)^n = 1 + {}^nC_1\, e^i\theta + {}^nC_2\, e^{2i}\theta + ... + {}^nC_n e^{in}\theta$

$\Rightarrow \quad (1 + \cos\theta + i\sin\theta)^n = 1 + {}^nC_1(\cos\theta + i\sin\theta)$

$+ {}^nC_2(\cos 2\theta + i\sin 2\theta) + ... + {}^nC_n(\cos n\theta + i\sin n\theta)$ ...(i)

L.H.S. $= \left(2\cos^2\frac{\theta}{2} + i \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$

$= \left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right)$

On equating real parts from both sides of (i), we get

$1 + {}^nC_1\cos\theta + {}^nC_2\cos 2\theta + ... + {}^nC_n\cos n\theta = \left(2\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}$

44. (c) : We have $\begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$

$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$

45. (c) : $\because B = \text{adj } A$

$\therefore \quad |B| = |\text{adj } A| \Rightarrow 64 = |A|^{3-1} \Rightarrow |A| = \pm 8$

46. (c) : $Q^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$Q^3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\therefore \quad Q^3x = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

47. (c) : Let $(x, x) \in Z$, then $x + 2x = 3x$ is divisible by 3

$\Rightarrow \quad xRx \Rightarrow R$ is reflexive

If $x + 2y$ is divisible by 3, then $y + 2x$ is also always divisible by 3 $\Rightarrow R$ is symmetric

Let xRy and yRz for some $x, y, z \in Z$

Let $x + 2y = 3p$ and $y + 2z = 3q$

$x + 3y + 2z = 3(p + q)$

$\Rightarrow \quad x + 2z = 3(p + q - y)$, which is divisible by 3

$\Rightarrow \quad xRz \Rightarrow R$ is transitive

$\therefore \quad R$ is an equivalence relation.

48. (c) : $A = \{0, 16, 112, ...\}$, $B = \{0, 16, 32, 48, ..., 112, ...\}$

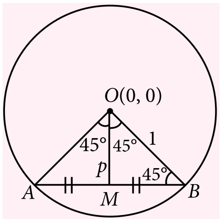
$\therefore \quad A \subseteq B$

49. (c) : $\because f(1) = 2^{35} = f(-1) \Rightarrow$ many-one

$\because \quad f(x)$ assumes only positive values as minimum value of $f(x) = 1 \Rightarrow$ not onto

50. (c) : If S.D. of $a_1, a_2, a_3, ..., a_n$ be σ, then S.D. of $\lambda a_1, \lambda a_2, \lambda a_3, ..., \lambda a_n$ will be $|\lambda|\sigma$.

51. (b) : $\because p^2 + p^2 = 1^2 \Rightarrow p^2 = \frac{1}{2}$



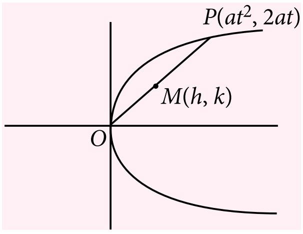
Let $M(h, k)$ be the mid point of chord AB.

$\therefore \quad OM = p \Rightarrow h^2 + k^2 = p^2$

$\Rightarrow \quad x^2 + y^2 = \frac{1}{2}$

is the required locus.

52. (d) : $y^2 = 4ax$



$h = \frac{at^2}{2}, \quad k = \frac{2at}{2}$

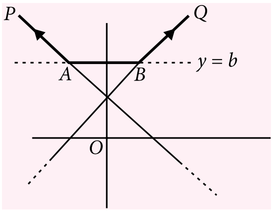
$\Rightarrow \quad 2h = a\left(\frac{k}{a}\right)^2$

$\Rightarrow \quad k^2 = 2ah$

$\Rightarrow \quad y^2 = 2ax$ is the locus. Its directrix is $x = -\frac{a}{2}$

53. (b) : $\int_0^2 x^2[x]\,dx = \int_0^1 x^2(0)dx + \int_1^2 x^2(1)dx = \frac{7}{3}$

54. (c) : Graph of $f(x) = \max\{a - x, a + x, b\}$ where $0 < a < b$ is given $(PABQ)$ As there are 2 steep corners (at A and B), $f(x)$ is not differentiable at these 2 points.



55. (d) : $|\vec{a} + \vec{b}|^2 < |\vec{a} - \vec{b}|^2$

$\Rightarrow \quad a^2 + b^2 + 2\vec{a}\cdot\vec{b} < a^2 + b^2 - 2\vec{a}\cdot\vec{b}$

$\Rightarrow \quad 4\vec{a}\cdot\vec{b} < 0 \Rightarrow \vec{a}\cdot\vec{b} < 0 \Rightarrow \cos\theta < 0$

$\therefore \quad \theta$ is an obtuse angle.

56. (b) : $\because \quad y\frac{dy}{dx} + by^2 = a\cos x$

$\therefore \quad 2y\frac{dy}{dx}\cdot e^{2bx} + y^2\cdot e^{2bx}\cdot 2b = 2a\cos x\cdot e^{2bx}$

[On multiplying by $2e^{2bx}$]

$\Rightarrow \quad \frac{d}{dx}(y^2\cdot e^{2bx}) = 2a\cos x\cdot e^{2bx}$

On integrating, $y^2\cdot e^{2bx} = 2a\int\cos x\cdot e^{2bx}dx$

$= 2a\cdot\frac{e^{2bx}(2b\cos x + \sin x)}{(2b)^2 + 1^2} + c'$

$\Rightarrow \quad (4b^2 + 1)y^2 = 2a(2b\cos x + \sin x) + c'(4b^2 + 1)e^{-2bx}$

$= 2a(2b\cos x + \sin x) + ce^{-2bx}$

57. (a) : $\because 16x^2 + 9y^2 = 400$ ...(i)

Given that $\frac{dy}{dt} = -\frac{dx}{dt}$ ...(ii)

Differentiating (i) w.r.t. t, we get

$16x\cdot\frac{dx}{dt} = 9y\cdot\frac{dx}{dt}$ [using (ii)] $\Rightarrow \quad x = \frac{9y}{16}$

From (i), $16\cdot\frac{81y^2}{16\times 16} + 9y^2 = 400$

$\Rightarrow \quad \frac{225y^2}{16} = 400 \Rightarrow y = \pm\frac{16}{3}$

When $y = \frac{16}{3}$, $x = 3$; when $y = -\frac{16}{3}$, $x = -3$

$\therefore \quad$ Required points are $\left(3, \frac{16}{3}\right)$ and $\left(-3, -\frac{16}{3}\right)$

58. (a) : Arranging in alphabetical order $\rightarrow$ C, C, H, I, N, O

CC .... $\rightarrow 4!$

CH .... $\rightarrow 4!$

CI .... $\rightarrow 4!$

CN .... $\rightarrow 4!$

COCHIN $\rightarrow 1$

$\therefore$ No. of words before COCHIN $= 4! + 4! + 4! + 4! = 96$

59. (d) : Let $A = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ x & 0 & x \end{pmatrix}$

$A^2 = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & x^2 & 0 \\ 2x^2 & 0 & x^2 \end{pmatrix}, A^3 = \begin{pmatrix} x^3 & 0 & 0 \\ 0 & x^3 & 0 \\ 3x^3 & 0 & x^3 \end{pmatrix}$

$\therefore \quad$ From principle of mathematical induction, we can say that $A^n = \begin{pmatrix} x^n & 0 & 0 \\ 0 & x^n & 0 \\ nx^n & 0 & x^n \end{pmatrix}$

So, when $x = 2$, $A^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ n\cdot 2^n & 0 & 2^n \end{pmatrix}$

$\therefore \quad a = 2^n, b = n\cdot 2^n$

60. (a) : On putting $n = 1, 2, 3$. Only option (a) is satisfied.

61. (a) : We have $\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

$\alpha^2 + \beta^2 = \left(-\frac{b}{a}\right)^2 - 2\cdot\frac{c}{a} = \frac{b^2 - 2ac}{a^2}$ and $\alpha^2\cdot\beta^2 = \frac{c^2}{a^2}$

$\therefore \quad$ Required equation is $x^2 - \left(\frac{b^2 - 2ac}{a^2}\right)x + \frac{c^2}{a^2} = 0$

$\Rightarrow \quad a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

62. (a, d) : $t_r = r\{(r + 1) - \omega\}\{(r + 1) - \omega^2\}$

$= r\{(r + 1)^2 - (r + 1)(\omega^2 + \omega) + \omega^3\}$

$= r\{(r + 1)^2 + (r + 1) + 1\} = r(r^2 + 3r + 3) = (r + 1)^3 - 1$

$\therefore \quad$ Required sum $= \sum_{r=1}^{n-1}\{(r + 1)^3 - 1\}$

$= (2^3 + 3^3 + 4^3 + \ldots + n^3) - 1\times(n - 1)$

$= (1^3 + 2^3 + 3^3 + \ldots + n^3) - n = \left\{\frac{n(n + 1)}{2}\right\}^2 - n$

63. (c) : ${}^nC_{r-1} = 36$ ...(i) ${}^nC_r = 84$ ...(ii) ${}^nC_{r+1} = 126$ ...(iii)

(i) ÷ (ii), we get $\dfrac{r}{n-r+1} = \dfrac{3}{7}$

$\Rightarrow$ $10r = 3n + 3$ ...(iv)

(ii) ÷ (iii), $\dfrac{r+1}{n-r} = \dfrac{2}{3} \Rightarrow 3r + 3 = 2n - 2r$

$\Rightarrow$ $5r = 2n - 3$ ...(v)

From (iv) and (v), we get $3n + 3 = 4n - 6$

$\Rightarrow$ $n = 9$. $\therefore$ ${}^nC_8 = {}^9C_8 = 9$.

64. (b) : $P(40/F) = \dfrac{P(40 \cap F)}{P(F)} = \dfrac{3/20}{6/20} = \dfrac{1}{2}$

65. (b) : We have $x^2(x - y) + (x - y) = 0$

$\Rightarrow$ $(x - y)(x^2 + 1) = 0$

$\Rightarrow$ $x - y = 0$ $[\because x^2 + 1 \neq 0]$ $\Rightarrow$ straight line

66. (*) : (Question appears to be wrong!)

[Instead of $(2n + 1)^{th}$ terms it should be $(2n - 1)^{th}$ terms]

$\because$ In any series of $(2n - 1)$ terms, the middle term is t_n. According to problem, t_n of A.P., G.P. and H.P. are a, b, c respectively. Hence, a, b, c are A.M., G.M. and H.M. respectively.

$\because$ A.M. $\geq$ G.M. $\geq$ H.M. $\Rightarrow$ $a \geq b \geq c$

Further, $(\text{G.M.})^2 = (\text{A.M.}) \times (\text{H.M.})$

$\therefore$ $b^2 = ac$ $\Rightarrow$ $ac - b^2 = 0$

$\therefore$ Options (b) and (d) should be correct.

67. (b, d) : Any point on $x + y + 1 = 0$ be $P(\alpha, -\alpha - 1)$

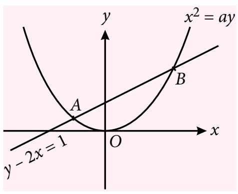
If its distance from $3x + 4y + 2 = 0$ is 1/5, then

$$\frac{3\alpha + 4(-\alpha - 1) + 2}{\sqrt{3^2 + 4^2}} = \pm\frac{1}{5}$$

$\Rightarrow$ $-\alpha - 2 = \pm 1$ $\Rightarrow$ $\alpha = -3 - 1$

$\Rightarrow$ P may be $(-3, 2)$ or $(-1, 0)$

68. (a, b): Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the points of intersection.



On solving, $x^2 = a(2x + 1)$

$\Rightarrow$ $x^2 - 2ax - a = 0$

$\therefore$ $x_1 + x_2 = 2a, x_1x_2 = -a$

$\because$ $AB = \sqrt{40}$

$\therefore$ $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{40}$

$\Rightarrow$ $\sqrt{(x_2 - x_1)^2 + \{2(x_2 - x_1)\}^2} = \sqrt{40}$

$[\because (x_1, y_1)$ lies on $y = 2x + 1$ $\therefore$ $y_1 = 2x_1 + 1]$

$\Rightarrow$ $5\{(x_2 - x_1)^2\} = 40 \Rightarrow (x_1 + x_2)^2 - 4x_1x_2 = 8$

$\Rightarrow 4a^2 + 4a = 8 \Rightarrow a^2 + a - 2 = 0$ $\therefore$ $a = 1, -2$

69. (b, c) : $\because f'(x) = (x - 1)^2(4 - x) = -x^3 + 6x^2 - 9x + 4$

On integrating, $f(x) = -\dfrac{x^4}{4} + 2x^3 - \dfrac{9x^2}{2} + 4x + C$

Putting $x = 0, f(0) = C$ (may or may not be zero)

$\because$ $f'(x) \geq 0$ in $(0, 3)$ $\therefore$ $f(x)$ is increasing in $(0, 3)$

$\because$ $f'(4) = 0$, $\therefore$ $x = 4$ is a critical point of $f(x)$

$\because$ $f'(x) > 0$ in $(3, 4)$ and $f'(x) < 0$ in $(4, 5)$

$\therefore$ We can't say that $f(x)$ is decreasing in $(3, 5)$.

70. (b, d) : We have, $4x^2 + 9y^2 = 1$ ...(i) $8x = 9y$ ...(ii)

Differentiating (i) w.r.t. x, we get

$8x + 18y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4x}{9y}$

$\Rightarrow$ slope of tangent $= \dfrac{-4x}{9y}$. Also, slope of line (ii) $= \dfrac{8}{9}$

Since line (ii) is parallel to the tangent $\therefore$ $\dfrac{-4x}{9y} = \dfrac{8}{9}$

$\Rightarrow$ $x = -2y$

From (i), $4(4y^2) + 9y^2 + 1 \Rightarrow y^2 = \dfrac{1}{25} \Rightarrow y = \pm\dfrac{1}{5}$

When $y = \dfrac{1}{5}$, $x = -\dfrac{2}{5}$; when $y = -\dfrac{1}{5}$, $x = \dfrac{2}{5}$

$\therefore$ Points are $\left(-\dfrac{2}{5}, \dfrac{1}{5}\right)$ and $\left(\dfrac{2}{5}, -\dfrac{1}{5}\right)$

71. (a, b) : Let $I = \displaystyle\int_{-3000}^{3000} \{\varphi(t - 2014) + \varphi(t - 2015)$

$+ \varphi(t - 2016)\}\varphi(t - 2016)dt$

$= \displaystyle\int_{-3000}^{2016} .... + \int_{2016}^{2017} .... + \int_{2017}^{3000} ....$

$= \displaystyle\int_{-3000}^{2016} [....] \cdot 0 \cdot dt + \int_{2016}^{2017} (0 + 0 + 1) \cdot 1 \cdot dt$

$+ \displaystyle\int_{2017}^{3000} (0 + 0 + 0) \cdot 0 \cdot dt$

$= 0 + 1 + 0 = 1$, which is real.

72. (a, c): $x^2 + y^2 - 10x + 21 = 0$

$x^2 - 10x + (y^2 + 21) = 0$

It has real roots if $D \geq 0$ $\Rightarrow$ $100 - 4(y^2 + 21) \geq 0$

$\Rightarrow$ $y^2 + 21 \leq 25$ $\Rightarrow$ $y^2 \leq 4$ $\Rightarrow$ $-2 \leq y \leq 2$

Also, $y^2 + (x^2 - 10x + 21) = 0$ will have real root if $D \geq 0$

$\Rightarrow$ $0 - 4(x^2 - 10x + 21) \geq 0$ $\Rightarrow$ $(x - 3)(x - 7) \leq 0$

$\Rightarrow$ $3 \leq x \leq 7$

73. (a, c) : $z = \sin\theta - i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) - i\sin\left(\dfrac{\pi}{2} - \theta\right)$

$\therefore$ $z^n = \cos\left(\dfrac{n\pi}{2} - n\theta\right) - i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$

$z^{-n} = \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$

$\Rightarrow$ $z^n + z^{-n} = 2\cos\left(\dfrac{n\pi}{2} - n\theta\right)$

and $z^n - z^{-n} = -2i\sin\left(\dfrac{n\pi}{2} - n\theta\right) = 2i\sin\left(n\theta - \dfrac{n\pi}{2}\right)$

74. (a, b) : $\because f(f(x)) = x$ $\Rightarrow f^{-1}(x) = f(x)$ $\Rightarrow f(x) = x$

$\therefore$ $f(x)$ is one-one and onto.

75. (a, c) : $P(A \cup B) \geq \dfrac{3}{4}$ ...(i) $\dfrac{1}{8} \leq P(A \cap B) \leq \dfrac{3}{8}$ ...(ii)

Let $P(A) + P(B)$ be x

From (i),

$x - P(A \cap B) \geq \dfrac{3}{4} \Rightarrow x - \dfrac{3}{4} \geq P(A \cap B) \geq \dfrac{1}{8}$ [From (ii)]

$\Rightarrow$ $x \geq \dfrac{7}{8}$ $\because$ $P(A \cup B) \leq 1 \Rightarrow x - P(A \cap B) \leq 1$ [From (ii)]

$\Rightarrow$ $x - 1 \leq P(A \cap B) \leq \dfrac{3}{8}$ $\Rightarrow$ $x \leq \dfrac{11}{8}$

PAPER-1

*ALOK KUMAR, B.Tech, IIT Kanpur

SECTION 1 (Maximum Marks : 15)

This section contains FIVE questions. Each question has FOUR options (a), (b), (c) and (d). ONLY ONE of these four options is correct. For each question, darken the bubble corresponding to the correct option in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened. Zero Marks : 0 If none of the bubbles is darkened. Negative Marks : –1 In all other cases.

1. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that P(computer turns out to be defective given that it is produced in plant T_1) = 10 P (computer turns out to be defective given that it is produced in plant T_2). A computer produced in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is

   (a) $\frac{36}{73}$ (b) $\frac{47}{79}$ (c) $\frac{78}{93}$ (d) $\frac{75}{83}$

2. Let $S=\left\{x\in(-\pi,\pi): x\neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x+\operatorname{cosec} x+2(\tan x-\cot x)=0$ in the set S is equal to

   (a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$ (c) 0 (d) $\frac{5\pi}{9}$

3. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is

   (a) 380 (b) 320 (c) 260 (d) 95

4. Let $-\frac{\pi}{6}<\theta<-\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x\sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x\tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

   (a) $2(\sec\theta - \tan\theta)$ (b) $2\sec\theta$
   (c) $-2\tan\theta$ (d) 0

5. The least value of $a \in R$ for which $4ax^2+\frac{1}{x}\geq 1$, for all $x > 0$, is

   (a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{27}$ (d) $\frac{1}{25}$

SECTION-2 (MAXIMUM MARKS : 32)

This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened. Partial Marks : +1 For darkening a bubble corresponding to each correct option, provided NO incorrect option is darkened. Zero Marks : 0 If none of the bubbles is darkened. Negative Marks : –2 In all other cases.

For example, if (a), (c) and (d) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (a) and (d) will result in +2 marks; and darkening (a) and (b) will result in –2 marks, as a wrong option is also darkened.

6. Consider a pyramid $OPQRS$ located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base $OPQR$ of the pyramid is a square with $OP = 3$. The point S is directly above the mid-point T of diagonal OQ such that $TS = 3$. Then

*Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91).
He trains IIT and Olympiad aspirants

(a) the acute angle between OQ and OS is $\frac{\pi}{3}$

(b) the equation of the plane containing the triangle OQS is $x - y = 0$

(c) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(d) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

7. The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(a) $Q_2Q_3 = 12$

(b) $R_2R_3 = 4\sqrt{6}$

(c) area of the triangle OR_2R_3 is $6\sqrt{2}$

(d) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

8. Let $f : (0, \infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(a) $\lim_{x \to 0^+} f'\left(\frac{1}{x}\right) = 1$

(b) $\lim_{x \to 0^+} xf\left(\frac{1}{x}\right) = 2$

(c) $\lim_{x \to 0^+} x^2 f'(x) = 0$

(d) $|f(x)| \le 2$ for all $x \in (0, 2)$

9. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

(a) $\alpha = 0, k = 8$

(b) $4\alpha - k + 8 = 0$

(c) $\det(P \operatorname{adj}(Q)) = 2^9$

(d) $\det(Q \operatorname{adj}(P)) = 2^{13}$

10. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. The the locus of E passes through the point(s)

(a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

11. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$ passes through the point $(1, 3)$. Then the solution curve

(a) intersects $y = x + 2$ exactly at one point

(b) intersects $y = x + 2$ exactly at two points

(c) intersects $y = (x + 2)^2$

(d) does not intersect $y = (x + 3)^2$

12. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

(a) area of the triangle XYZ is $6\sqrt{6}$

(b) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(c) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(d) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

13. Let $f : R \to R, g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2, g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$

(c) $h(0) = 16$ (d) $h(g(3)) = 36$

SECTION 3 (MAXIMUM MARKS : 15)

This section contains FIVE questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened. Zero Marks : 0 In all other cases.

14. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\,dt = 2x - 1$ is

15. Let $\alpha, \beta \in R$ be such that $\lim_{x\to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals

16. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + ... + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is

17. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

18. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

PAPER-2

SECTION 1 (Maximum Marks : 18)

This section contains SIX questions. Each question has FOUR options (a), (b), (c) and (d). ONLY ONE of these four options is correct. For each question, darken the bubble corresponding to the correct option in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened. Zero Marks : 0 If none of the bubbles is darkened. Negative Marks : –1 In all other cases.

1. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals

(a) 52 (b) 103 (c) 201 (d) 205

2. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

(a) $x + y - 3z = 0$ (b) $3x + z = 0$
(c) $x - 4y + 7z = 0$ (d) $2x - y = 0$

3. Area of the region $\left\{(x, y) \in R^2 : y \ge \sqrt{|x+3|},\ 5y \le x+9 \le 15\right\}$ is equal to

(a) 1/6 (b) 4/3 (c) 3/2 (d) 5/3

4. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

(a) $3-\sqrt{3}$ (b) $2(3-\sqrt{3})$
(c) $2(\sqrt{3}-1)$ (d) $2(2+\sqrt{3})$

5. Let $b_i > 1$ for $i = 1, 2, ..., 101$. Suppose $\log_e b_1, \log_e b_2, ..., \log_e b_{101}$ are in arithmetic progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then

(a) $s > t$ and $a_{101} > b_{101}$
(b) $s > t$ and $a_{101} < b_{101}$
(c) $s < t$ and $a_{101} > b_{101}$
(d) $s < t$ and $a_{101} < b_{101}$

6. The value of $\int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x}\,dx$ is equal to

(a) $\frac{\pi^2}{4}-2$ (b) $\frac{\pi^2}{4}+2$
(c) $\pi^2 - e^{\pi/2}$ (d) $\pi^2 + e^{\pi/2}$

SECTION-2 (Maximum Marks : 32)

This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened. Partial Marks : +1 For darkening a bubble corresponding to each correct option, provided NO incorrect option is darkened. Zero Marks : 0 If none of the bubbles is darkened. Negative Marks : –2 In all other cases.

For example, if (a), (c) and (d) are all the correct options for a question, darkening all these three will result in +4 marks; darkening only (a) and (d) will result in +2 marks; and darkening (a) and (b) will result in –2 marks, as a wrong option is also darkened.

7. Let $f:\left[-\frac{1}{2}, 2\right]\to R$ and $g:\left[-\frac{1}{2}, 2\right]\to R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|\,f(x) + |4x - 7|\,f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then
   (a) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$
   (b) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$
   (c) g is not differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$
   (d) g is not differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

8. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\hat{u}\times\vec{v}| = 1$ and $\hat{w}\cdot(\hat{u}\times\vec{v}) = 1$ Which of the following statement(s) is(are) correct?
   (a) There is exactly one choice for such $\vec{v}$.
   (b) There are infinitely many choices for such $\vec{v}$.
   (c) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$.
   (d) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$.

9. Let $a, b \in R$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in C : z = \frac{1}{a+ibt},\ t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on
   (a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$
   (b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$.
   (c) the x-axis for $a \neq 0, b = 0$
   (d) the y-axis for $a = 0, b \neq 0$.

10. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then
   (a) $SP = 2\sqrt{5}$
   (b) $SQ : QP = (\sqrt{5}+1) : 2$
   (c) the x-intercept of the normal to the parabola at P is 6
   (d) the slope of the tangent to the circle at Q is $\frac{1}{2}$

11. Let $f(x) = \lim\limits_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\left(x+\frac{n}{n}\right)}{n!\,(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then
   (a) $f\left(\frac{1}{2}\right) \geq f(1)$
   (b) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$
   (c) $f'(2) \leq 0$
   (d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

12. Let $a, b \in R$ and $f : R \to R$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is
   (a) differentiable at $x = 0$ if $a = 0$ and $b = 1$
   (b) differentiable at $x = 1$ if $a = 1$ and $b = 0$
   (c) not differentiable at $x = 0$ if $a = 1$ and $b = 0$
   (d) not differentiable at $x = 1$ if $a = 1$ and $b = 1$

13. Let $f : R \to (0, \infty)$ and $g : R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then
   (a) f has a local minimum at $x = 2$
   (b) f has a local maximum at $x = 2$
   (c) $f''(2) > f(2)$
   (d) $f(x) - f''(x) = 0$ for at least one $x \in R$

14. Let $a, \lambda, \mu \in R$. Consider the system of linear equations $ax + 2y = \lambda$, $3x - 2y = \mu$. Which of the following statement(s) is(are) correct?
   (a) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ.
   (b) If $a \neq -3$, then the system has a unique solution for all values of λ and μ.

(c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$.

(d) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$.

SECTION-3 (Maximum Marks : 12)

This section contains TWO paragraphs. Based on each paragraph, there are TWO questions. Each question has four options (a), (b), (c) and (d). ONLY ONE of these four options is correct. For each question, darken the bubble corresponding to the correct option in the ORS. For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened. Zero Marks : 0 In all other cases.

Paragraph-1

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2 respectively, after two games.

15. $P(X > Y)$ is

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{7}{12}$

16. $P(X = Y)$ is

(a) $\frac{11}{36}$ (b) $\frac{1}{3}$ (c) $\frac{13}{36}$ (d) $\frac{1}{2}$

Paragraph-2

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$, for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

17. The orthocentre of the triangle F_1MN is

(a) $\left(-\frac{9}{10}, 0\right)$ (b) $\left(\frac{2}{3}, 0\right)$

(c) $\left(\frac{9}{10}, 0\right)$ (d) $\left(\frac{2}{3}, \sqrt{6}\right)$

18. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is

(a) 3 : 4 (b) 4 : 5 (c) 5 : 8 (d) 2 : 3

SOLUTIONS

PAPER-1

1. (c): Let $\lambda = P$ (computer turns out to be defective given that it is produced in plant T_2) $= P(D/T_2)$

Then $P(D/T_1) = 10\lambda$

Also, $P(D)=\frac{7}{100}$

Using theorem on total probability,

$$P(D) = P(T_1)\cdot P(D/T_1) + P(T_2)\cdot P(D/T_2)$$

$$\Rightarrow \frac{7}{100}=\frac{20}{100}\cdot 10\lambda+\frac{80}{100}\cdot\lambda$$

On solving, we get $\lambda=\frac{1}{40}$

We want $P(T_2/\bar{D})$

We have $P(\bar{D}/T_1)=1-P(D/T_1)=1-\frac{10}{40}=\frac{30}{40}$

and $P(\bar{D}/T_2)=1-P(D/T_2)=1-\frac{1}{40}=\frac{39}{40}$

Then by Baye's theorem

$$P(T_2/\bar{D})=\frac{\frac{39}{40}\cdot\frac{80}{100}}{\frac{39}{40}\cdot\frac{80}{100}+\frac{30}{40}\cdot\frac{20}{100}}=\frac{1}{1+\frac{30\cdot 20}{39\cdot 80}}$$

$$=\frac{1}{1+\frac{15}{78}}=\frac{78}{93}$$

[Rating : Easy]

2. (c): On simplification, we have

$$\sqrt{3}\sin x+\cos x=2\cos 2x$$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x=\cos 2x$$

Notice that $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3}=\frac{1}{2}$, $\therefore$ We have

$$\cos\left(x-\frac{\pi}{3}\right)=\cos 2x$$

We have, $2x\pm\left(x-\frac{\pi}{3}\right)=2k\pi, k\in Z$

Then $x=(6k-1)\frac{\pi}{3}$ or $(6k+1)\frac{\pi}{9}, k\in Z$

The values of x in $(-\pi, \pi)$ are

$$-\frac{\pi}{3}, \frac{\pi}{9}, \frac{7\pi}{9}, -\frac{5\pi}{9}$$

Their sum turns to be zero.

[Rating : Easy]

3. (a) : The team can have either 0 boy or 1 boy. So the number of selection is $({}^6C_4\cdot{}^4C_0+{}^6C_3\cdot{}^4C_1)\cdot{}^4C_1$

$= (15 + 20 \cdot 4) \cdot 4 = 95 \cdot 4 = 380$

[Rating : Easy]

4. **(c):** $x^2 - 2x\sec\theta + 1 = 0$ gives

$$x = \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2} = \sec\theta \pm |\tan\theta|$$

As $\theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)$, we have $\tan\theta < 0$

Thus the large root α_1 is given by $\alpha_1 = \sec\theta - \tan\theta$

Again, $x^2 + 2x\tan\theta - 1 = 0$ gives

$$x = \frac{-2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2} = -\tan\theta \pm |\sec\theta|$$

The small root β_2 is given by $\beta_2 = -\tan\theta - \sec\theta$

Thus, $\alpha_1 + \beta_2 = -2\tan\theta$

[Rating : Easy]

5. **(c):** Let $f(x) = 4ax^2 + \frac{1}{x} \quad (x > 0)$

Now, $f'(x) = 8ax - \frac{1}{x^2}$

f attains its minimum at $x_0 = \left(\frac{1}{8a}\right)^{1/3}$

As $f(x) \geq 1 \ \forall \ x > 0$

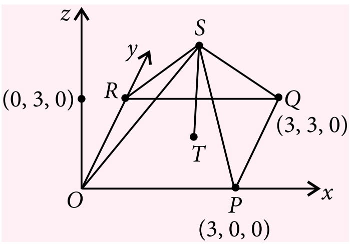
Then $f(x_0) \geq 1$

$$\Rightarrow \quad 4ax_0^2 + \frac{1}{x_0} \geq 1 \ \Rightarrow \ 4ax_0^3 + 1 \geq x_0$$

$$\Rightarrow \quad 4a \cdot \frac{1}{8a} + 1 \geq \left(\frac{1}{8a}\right)^{1/3} \ \Rightarrow \ \frac{27}{8} \geq \frac{1}{8a} \ \therefore \ a \geq \frac{1}{27}$$

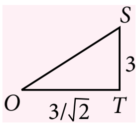
[Rating : Doable]

6. **(b, c, d) :**



$$T \equiv \left(\frac{3}{2}, \frac{3}{2}, 0\right), \ S \equiv \left(\frac{3}{2}, \frac{3}{2}, 3\right)$$

From ΔOTS,



$$\tan\angle SOT = \frac{3}{3/\sqrt{2}} = \sqrt{2}$$

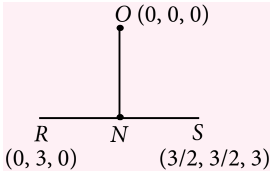
$\therefore$ The angle between OQ and OS is $\tan^{-1}\sqrt{2}$ (not $\pi/3$)

The equation of plane containing $O(0, 0, 0)$, $Q(3, 3, 0)$ and $S\left(\frac{3}{2}, \frac{3}{2}, 3\right)$ is $x = y$

The length of perpendicular from point P to plane OQS is equal to $PT = \frac{3}{\sqrt{2}}$

Let N be the foot of the perpendicular from O to RS.

$$N \equiv \left(\frac{3\lambda}{2}, 3 - \frac{3\lambda}{2}, 3\lambda\right)$$



As $ON \perp RS$, we have

$$\frac{9}{4}\lambda - \frac{3}{2}\left(3 - \frac{3\lambda}{2}\right) + 9\lambda = 0$$

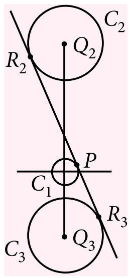
$$\Rightarrow \lambda = \frac{1}{3}$$

$\therefore$ The point N is $\left(\frac{1}{2}, \frac{5}{2}, 1\right)$

Also, $ON = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + 1^2} = \sqrt{\frac{30}{4}} = \sqrt{\frac{15}{2}}$

[Rating : Doable]

7. **(a, b, c) :** The point P is given by solving $x^2 + y^2 = 3$ with $x^2 = 2y$ *i.e.,*



$y^2 + 2y = 3 \quad \Rightarrow \quad y^2 + 2y - 3 = 0$

$\Rightarrow \quad (y - 1)(y + 3) = 0$

$\because \quad P$ is in the first quadrant, so $y = 1$

$\therefore \quad P(\sqrt{2}, 1)$

The points Q_2 and Q_3 are given by $Q_2 = (0, 9)$ and $Q_3 = (0, -3)$

$\therefore \quad Q_2Q_3 = \sqrt{0^2 + 12^2} = 12$

Again, $R_2R_3 = \sqrt{(Q_2Q_3)^2 - (r_2 + r_3)^2} \quad (r_2 = r_3 = 2\sqrt{3})$

$$= \sqrt{12^2 - (4\sqrt{3})^2} = \sqrt{144 - 48} = \sqrt{96} = 4\sqrt{6}$$

The distance of origin from $R_2R_3 = \sqrt{3}$

$$\therefore \ [OR_2R_3] = \frac{1}{2} \cdot 4\sqrt{6} \cdot \sqrt{3} = 6\sqrt{2}$$

Again the distance P from $Q_2Q_3 = \sqrt{2}$

$$[PQ_2Q_3] = \frac{1}{2} \cdot 12 \cdot \sqrt{2} = 6\sqrt{2}$$

[Rating : Difficult]

8. **(a) :** $f'(x) = 2 - \frac{f(x)}{x}$

$\Rightarrow \quad f'(x) + \frac{1}{x}f(x) = 2$ is linear differential equation.

Hence, $I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Thus the solution is given by

$$x \cdot f(x) = \int 2x\,dx + \lambda \ \textit{i.e.} \ x f(x) = x^2 + \lambda$$

As $f(1) \neq 1$, we have $\lambda \neq 0$

$$\therefore \ f(x) = x + \frac{\lambda}{x}, \ \lambda \neq 0$$

Thus, $f'(x) = 1 - \frac{\lambda}{x^2}, \ \lambda \neq 0$

Now, $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+} (1 - \lambda x^2) = 1$

$\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x} + \lambda x\right) = \lim_{x\to 0^+} (1 + \lambda x^2) = 1$

$\lim_{x\to 0^+} x^2 f'(x) = \lim_{x\to 0^+} x^2\left(1 - \frac{\lambda}{x^2}\right) = \lim_{x\to 0^+} (x^2 - \lambda) = -\lambda$

Again, $\lim_{x\to 0^+} f(x) \to \infty$

Hence the function is not bounded.
Note that λ can be –ve or +ve.

[Rating : Difficult]

9. (b, c): $PQ = kI$

$\Rightarrow \quad Q = kP^{-1}I = \frac{k}{\det P}(\text{adj } P)I$

$$= \frac{k}{20+12\alpha}\begin{bmatrix} 5\alpha & 10 & -\alpha \\ 3\alpha & 6 & -(3\alpha+4) \\ -10 & 12 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As $q_{23} = -k/8$, we have $\frac{k}{20+12\alpha}\cdot(3\alpha+4) = \frac{k}{8}$

As $k \neq 0$, we have $2(3\alpha + 4) = 5 + 3\alpha$

$\Rightarrow \ 6\alpha + 8 = 5 + 3\alpha \quad \therefore \quad \alpha = -1$

As $\det Q = \frac{k^3}{\det P}$ we have $\frac{k^2}{2} = \frac{k^3}{20+12\alpha}$

$\Rightarrow \ 2k = 20 + 12\alpha \quad \therefore \quad k = 4$

$\det(P \text{ adj } Q) = \det(P)(\det \text{adj} Q)$

$$= 2k\cdot\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = \frac{2^{10}}{2} = 2^9.$$

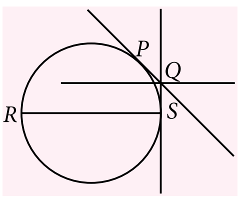
[Rating : Difficult]

10. (a, c): If $P(\cos\theta, \sin\theta)$ be the variable point.
The tangent at P is

$x\cos\theta + y\sin\theta = 1$

Tangent at S is $x = 1$

Thus Q is $\left(1, \frac{1-\cos\theta}{\sin\theta}\right)$



A line through Q parallel to SR is $y = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$

The normal at P is $y = x\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}x$

We have $h = \frac{1-\tan^2(\theta/2)}{2}, \ k = \tan\frac{\theta}{2}$

The locus is $h = \frac{1-k^2}{2} \ \Rightarrow \ k^2 = 1 - 2h$

i.e. $y^2 = 1 - 2x$

[Rating : Doable]

11. (a, d) : Put $x + 2 = u$ and $y = v$ leads to

$(u^2 + uv)\frac{dv}{du} = v^2$

$\Rightarrow \ (u^2 + uv)dv = v^2du$

$\Rightarrow \ u^2dv = v(vdu - udv)$

We have $\frac{dv}{v} = \frac{vdu - udv}{u^2}$

On integrating, we get

$\ln|v| = -\frac{v}{u} + \lambda$

As the curve passes through (1, 3), we have

$\lambda = 1 + \ln 3$

Then the curve is

$\frac{y}{x+2} + \ln|y| - 1 - \ln 3 = 0, \ x > 0$

Substitute $y = x + 2$ in the equation of curve we have

$1 + \ln|x + 2| - 1 - \ln 3 = 0$

$\therefore \ x = 1, -5$

The curve intersects $y = x + 2$ at point (1, 3).

Put $y = (x + 2)^2$ in the equation of the curve to get

$(x + 2) + 2\ln(x + 2) = 1 + \ln 3$

As the L.H.S. is an increasing function, hence it is greater than 2 + 2 ln2. Thus no solution.

Now put $y = (x + 3)^2$ in equation of curve,

$\frac{(x+3)^2}{x+2} + \ln(x+3)^2 - 1 - \ln 3 = 0$

As $x > 0$, we have $x + 3 > x + 2$ *i.e.* $x + 3 > 3$

$\frac{(x+3)^2}{x+2} + \frac{\ln(x+3)^2}{3} > 1$

Hence again there is no solution.

Thus, the curve $y = (x + 3)^2$ doesn't intersect the original curve.

[Rating : Difficult]

12. (a, c, d) : $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s-(x+y+z)}{9} = \frac{s}{9}$

$\Rightarrow \ \frac{x}{5} = \frac{y}{6} = \frac{z}{7} = \frac{x+y+z}{18} = \frac{s}{9} = \lambda$ (say)

Radius of incircle = $\sqrt{8/3}$

Now, $r=\frac{\Delta}{s} \Rightarrow \sqrt{\frac{8}{3}}=\frac{\sqrt{9\cdot 4\cdot 3\cdot 2}}{9}\lambda=\frac{6\sqrt{6}}{9}\lambda$

$\therefore\ \lambda=\frac{9}{6}\sqrt{\frac{8}{3}\times\frac{1}{6}}=\frac{3}{2}\sqrt{\frac{4}{9}}=\frac{3}{2}\cdot\frac{2}{3}=1$

$\therefore\ x=5, y=6, z=7$

Now, $\Delta=6\sqrt{6}\lambda^2=6\sqrt{6}$

$$R=\frac{xyz}{4\Delta}=\frac{5\cdot 6\cdot 7}{4\cdot 6\sqrt{6}}=\frac{35}{4\sqrt{6}}$$

$$\sin\frac{X}{2}\cdot\sin\frac{Y}{2}\cdot\sin\frac{Z}{2}=\frac{r}{4R}=\frac{\sqrt{\frac{8}{3}}}{4\times 35}\times 4\sqrt{6}=\frac{4}{35}$$

Again, $\sin^2\left(\frac{X+Y}{2}\right)=\cos^2\frac{Z}{2}=\frac{1+\cos Z}{2}$

$$=\frac{1+\frac{x^2+y^2-z^2}{2xy}}{2}=\frac{1+\frac{25+36-49}{2\cdot 5\cdot 6}}{2}$$

$$=\frac{1+\frac{1}{5}}{2}=\frac{3}{5}$$

[Rating : Medium]

13. (b, c): $f(x)=x^3+3x+2$, $g(f(x))=x$, $h(g(g(x)))=x$

Now, $f'(x)=3x^2+3$

Again, $g'(2)=\frac{1}{f'(0)}=\frac{1}{3}$ (As we have $f(0)=2$)

Now, $h(g(g(x)))=x$

Differentiating with respect to x,

$$h'(g(g(x)))=\frac{1}{g'(g(x))\,g'(x)}$$

Now, to solve $g(g(x))=1$ we have $g(x)=f(1)=6$

$f(6)=236$

$$\therefore\ h'(1)=\frac{1}{g'(6)g'(236)}=\frac{1}{\frac{1}{6}\cdot\frac{1}{111}}=666$$

Solving, $g(g(x))=0$ means $g(x)=g^{-1}(0)$.

$\Rightarrow\ g(x)=2\quad\therefore\ x=g^{-1}(2)=f(2)=16$

$\therefore\ h(0)=16$

Again, $h(g(g(x)))=x$

Put x to $f(x)$ then $h(g(g(f(x))))=f(x)$

$\Rightarrow\ h(g(x))=f(x)$

$\therefore\ h(g(3))=f(3)=38$

[Rating : Difficult]

14. (1) : $\int_0^x\frac{t^2}{1+t^4}dt=2x-1$

Let $g(x)=\int_0^x\frac{t^2}{1+t^4}dt-2x+1$

Now, $g'(x)=\frac{x^2}{1+x^4}-2=\frac{1}{x^2+\frac{1}{x^2}}-2$

$$\le\frac{1}{2}-2$$

As $g'(x)$ is –ve. $\therefore$ g is decreasing.

Also, $g(0)=1$ and $g(1)=\int_0^1\frac{t^2}{1+t^4}dt-1<0$

Thus, $g(x)=0$ possesses exactly one solution in [0, 1].

[Rating : Difficult]

15 (7) : Given $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=1$

If $\alpha\ne 1$, then $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{\alpha x-\sin x}=\lim_{x\to 0}\frac{x\sin(\beta x)}{\alpha-\frac{\sin x}{x}}=0$

$\therefore\ \alpha=1$

Then $\lim_{x\to 0}\frac{x^2\sin(\beta x)}{x-\sin x}=1\ \Rightarrow\ \lim_{x\to 0}\frac{\beta x^3\cdot\frac{\sin(\beta x)}{\beta x}}{x^3\left(\frac{x-\sin x}{x^3}\right)}=1$

$$\left[\text{Recall, }\lim_{x\to 0}\frac{x-\sin x}{x^3}=\frac{1-\cos x}{3x^2}=\frac{\sin x}{6x}=\frac{1}{6}\right]$$

$\Rightarrow\ 6\beta=1$

Thus, $6\alpha+6\beta=7$

[Rating : Easy]

16. (5) : The coefficient of x^2 in the expansion of $(1+x)^2+(1+x)^3+\ldots+(1+mx)^{50}$ is

${}^2C_2+{}^3C_2+\ldots+{}^{49}C_2+{}^{50}C_2\,m^2$

$={}^{50}C_3+{}^{50}C_2\,m^2$

It is given to be $(3n+1)\,{}^{51}C_3$

Now, $\frac{50\cdot 49\cdot 48}{6}+\frac{50\cdot 49}{2}m^2=(3n+1)\frac{51\cdot 50\cdot 49}{6}$

$\Rightarrow\ 51n+1=m^2$

As $51n+1$ must be perfect square.

So, we have by inspection, $n=5$. $\therefore\ m=16$

[Rating : Medium]

17. (2) : $\begin{vmatrix}x & x^2 & 1+x^3\\ 2x & 4x^2 & 1+8x^3\\ 3x & 9x^2 & 1+27x^3\end{vmatrix}=10$

$$\Rightarrow \quad x^3\begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

$$\Rightarrow \quad x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$$

$\Rightarrow \; x^3 \cdot 2 + x^6 \cdot 12 = 10$

$\Rightarrow \; 6x^6 + x^3 - 5 = 0 \quad \Rightarrow \quad (x^3 + 1)(6x^3 - 5) = 0$

Therefore two real solutions. **[Rating : Difficult]**

18. (1) : Note that $z = \omega$

$$P \equiv \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}\{(-1)^r + 1\} \\ \omega^{r+2s}\{(-1)^r + 1\} & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$$

$\Rightarrow \; (-1)^r + 1 = 0 \quad \therefore \quad r = 1$ or 3

Again, $\omega^{2r} + \omega^{4s} = -1$

r and s both can't be 3.

$\therefore \; r = 1$, then $s = 1$

$\therefore \; (r, s) = (1, 1)$ only. **[Rating : Difficult]**

Paper-2

1. (b) : $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$,

We have $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16(1+2) & 8 & 1 \end{bmatrix}$

Again, $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16(1+2+3) & 12 & 1 \end{bmatrix}$

Then by induction,

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ \dfrac{16 \cdot 50 \cdot 51}{2} & 200 & 1 \end{bmatrix}$$

As $P^{50} - Q = I$. We have $q_{31} = \dfrac{16 \cdot 50 \cdot 51}{2}$

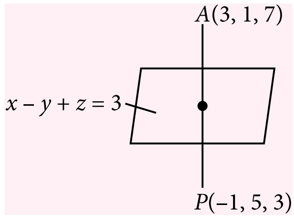
Again, $q_{32} = 200$, $q_{21} = 200$

$$\therefore \quad \frac{q_{31} + q_{32}}{q_{21}} = \frac{16 \cdot 50 \cdot 51}{2 \cdot 200} + 1 = 102 + 1 = 103$$

[Rating : Difficult]

2. (c): We have

$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \frac{-2(6)}{3} = -4$$



$\therefore \; x = -1, y = 5, z = 3$

The point P is $(-1, 5, 3)$

Now the plane containing P is

$\lambda(x + 1) + \mu(y - 5) + \upsilon(z - 3) = 0$

We have,

$\lambda + 2\mu + \upsilon = 0$ and

$\lambda - 5\mu - 3\upsilon = 0$

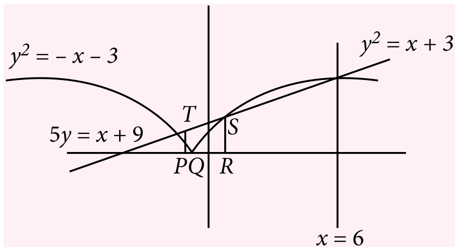
$\therefore \; \dfrac{\lambda}{-1} = \dfrac{\mu}{4} = \dfrac{\upsilon}{-7}$ (by cross multiplication)

Hence the equation of the plane is

$x - 4y + 7z = 0$.

[Rating : Easy]

3. (c): The graph of the region is sketched below.



Note that $y \geq \sqrt{|x+3|}$ means $y^2 \geq |x + 3|$

i.e. $y^2 \geq x + 3$ or $y^2 \geq -x - 3$

$P \equiv (-4, 0)$, $Q(-3, 0)$, $R(1, 0)$, $S(1, 2)$, $T(-4, 1)$

$$[PQT] = \int_{-4}^{-3} \sqrt{-x-3}\, dx = \int_0^1 \sqrt{t}\, dt = \left.\frac{t^{3/2}}{3/2}\right|_0^1 = \frac{2}{3}$$

$$\text{Again, } [QRS] = \int_{-3}^{1} \sqrt{x+3}\, dx = \int_0^4 \sqrt{u}\, du = \left.\frac{2}{3}u^{3/2}\right|_0^4 = \frac{16}{3}$$

$$[PRST] = \frac{1}{2}(1+2) \cdot 5 = \frac{15}{2}$$

The desired area $= \dfrac{15}{2} - \dfrac{16}{3} - \dfrac{2}{3} = \dfrac{3}{2}$

4.	**(c)**	**5.**	**(b)**	**6.**	**(a)**
7.	**(b, c)**	**8.**	**(b, c)**	**9.**	**(a, c, d)**
10.	**(a, c, d)**	**11.**	**(b, c)**	**12.**	**(a, b)**
13.	**(a, d)**	**14.**	**(b, c, d)**	**15.**	**(b)**
16.	**(c)**	**17.**	**(a)**	**18.**	**(c)**

For detailed solutions refer to : MTG JEE Advanced Explorer

◆◆

Rvp